Estimation de normes dans les espaces Lp non commutatifs et applications

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Estimations de normes dans les espaces L^p non commutatifs et applications

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Résumé

Résumé

Cette thèse présente quelques résultats d'analyse sur les espaces L^p le plus souvent non commutatifs. La première partie exhibe de large classes de contractions sur des espaces L^p non commutatifs qui vérifient l'analogue non commutatif de la conjecture de Matsaev. De plus, cette partie fournit une comparaison entre certaines normes apparaissant naturellement dans ce domaine. La deuxième partie traite des fonctions carrées. Le premier résultat principal énonce que si T est un opérateur R-Ritt sur un espace L^p alors les fonctions carrées associées sont équivalentes. Le second résultat principal est une caractérisation de certaines estimations carrées utilisant les dilatations. La troisième partie de cette thèse introduit de nouvelles fonctions carrées pour les opérateurs de Ritt définis sur des espaces L^p non commutatifs. Le résultat principal est qu'en général ces fonctions carrées ne sont pas équivalentes. Cette partie contient aussi un résultat d'équivalence entre la norme usuelle et une certaine fonction carrée. La quatrième partie introduit un analogue non commutatif de l'algèbre de Figà-Talamanca-Herz $A_p(G)$ sur le prédual naturel de l'espace d'opérateurs $\mathfrak{M}_{p,cb}$ des multiplicateurs de Schur complètement bornées sur l'espace de Schatten S^p.

Mots-clefs

Espaces L^p non commutatifs, espaces de Schatten, conjecture de Matsaev, multiplicateurs de Schur, dilatations, fonctions carrées, opérateurs de Ritt, algèbres de Figà-Talamanca-Herz, espaces d'opérateurs.

Mathematics Subject Classification (2010)

46B70, 46H99, 46L07, 46L51, 47A20, 47A60, 47A63, 47B38, 47L25.

Estimates of norms in noncommutative L^p-spaces and applications

Abstract

This thesis presents some results of analysis in L^p-spaces, especially often noncommutative. The first part exhibits large classes of contractions on noncommutative L^p-spaces which satisfy the noncommutative analogue of Matsaev's conjecture. Moreover, this part gives a comparison between various norms arising naturally from this field. The second part is devoted to square functions. The first main result states that if T is an R-Ritt operator on a L^p-space then the involved square functions are equivalent. The second principal result is a characterization of some square functions estimates in terms of dilations. In the third part of this thesis, we introduce some new square functions for Ritt operators defined on noncommutative L^p-spaces. The main result is that these square functions are generally not equivalent. This part also contains a result of equivalence between the usual norm and some special square function. The fourth part introduces a noncommutative analogue of the Figà-Talamanca-Herz algebra $A_p(G)$ on the natural predual of the operator space $\mathfrak{M}_{p,cb}$ of completely bounded Schur multipliers on the Schatten space S^p.

Keywords

Non-commutative L^p spaces, Schatten spaces, Matsaev's conjecture, Schur multipliers, dilations, square functions, Ritt operators, Figà-Talamanca-Herz algebras, operator spaces.

Mathematics Subject Classification (2010)

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Introduction générale

1 Introduction

Un des plus importants changements de point de vue de notre compréhension du monde physique est sans doute apparu avec les travaux de W. Heisenberg. Ce dernier a montré que l'on peut davantage comprendre la nature en substituant des matrices aux fonctions dans les théories mathématiques de celle-ci. Contrairement aux fonctions, les matrices ne commutent pas en général. Cette 'mécanique matricielle' a attiré un certain nombre de mathématiciens qui ont développé les mathématiques nécessaires à ce changement de paradigme. Cette démarche générale et importante a donné naissance aux *mathématiques non commutatives*.

Deux d'entre eux, F. Murray et J. von Neumann ont alors développé une théorie de l'intégration non commutative dans une remarquable série d'articles. Ils ont remplacé les algèbres de fonctions qui apparaissent naturellement dans la théorie de l'intégration classique par une certaine classe d'algèbres d'opérateurs bornées sur un espace de Hilbert. Plus tard, on a nommé ces dernières, *algèbres de von Neumann*.

A la suite de ces travaux et sous l'impulsion de J. Dixmier, I. Segal et d'autres, une théorie des espaces L^p non commutatifs se développa naturellement et rapidement. Récemment, les techniques matricielles et la théorie des espaces d'opérateurs ont redynamisé ce domaine.

Dans cette théorie, on utilise une trace τ sur une algèbre de von Neumann M à la place de l'intégrale. Les espaces L^p non commutatifs les plus simples sont les classes de Schatten, qui correspondent au cas où M est l'algèbre $B(\ell^2)$ de tous les opérateurs bornés sur l'espace de Hilbert ℓ^2 et où τ est la trace usuelle Tr. Les espaces L^p associés sont alors notés S^p et chacun d'entre eux s'identifie à l'ensemble des opérateurs x de $B(\ell^2)$ tels que $\|x\|_{S^p} = \big(\mathrm{Tr}\,(x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$. En utilisant une base orthonormale de ℓ^2, on peut voir ces derniers comme des matrices infinies. Ces espaces sont intensivement utilisés dans cette thèse. Notons au passage qu'on dit qu'une application linéaire T entre espaces L^p non commutatifs est *complètement bornée* si $T \otimes Id_{S^p}$ est bornée. Autrement dit, une application complètement bornée est une application qui reste bornée quand on remplace les scalaires par les matrices de S^p. Ces applications sont les applications naturelles de la théorie des espaces d'opérateurs.

Cette théorie se propose premièrement de développer des concepts similaires à ceux de la théorie des espaces L^p classiques, i.e *commutatifs*. Le chapitre 4 illustre bien ce point. Souvent, les deux situations se comportent de manière similaire comme par exemple dans le chapitre 2. Cependant, le cas non commutatif peut se comporter de manière très différente du cas classique amenant des constructions mathématiques nouvelles. La comparaison des résultats de dilatation du chapitre 1 avec le cas commutatif permet de bien comprendre cette particularité. De plus, dans ce contexte, il est parfois assez instructif de comparer directement les situations commutatives et non commutatives comme dans la section 3 du chapitre 1. Enfin, cette théorie apporte aussi des problèmes d'un type nouveau sans équivalent dans le cas commutatif. Le chapitre 3 de cette thèse illustre parfaitement ce fait.

2 Contenu de la thèse

Cette thèse est constituée de quatre chapitres, rédigés en anglais. Le premier chapitre de la thèse présente un article intitulé 'On Matsaev's conjecture for contractions on noncommutative L^p-spaces'. Ce texte a été accepté dans *Journal of Operator Theory*. Le second chapitre est un travail en collaboration avec C. Le Merdy intitulé 'Dilation of Ritt operators on L^p-spaces'. Le troisième chapitre intitulé 'Square functions for Ritt operators on noncommutative L^p-spaces' a été chronologiquement le dernier travail de cette thèse. Le dernier chapitre s'intitule 'Noncommutative Figà-Talamanca-Herz algebras for Schur multipliers' et a été publié dans *Integral Equations and Operator Theory* en 2011. Le reste de cette section est constituée des descriptions détaillées de chaque chapitre (chacun d'eux commence par une description analogue en anglais).

2.1 Chapitre 1

Estimer les normes des fonctions d'opérateurs est une tâche essentielle dans la théorie des opérateurs. Dans ce domaine, V. V. Matsaev a énoncé la conjecture suivante en 1971, voir [81]. Pour tout $1 \leqslant p \leqslant \infty$, désignons par $S : \ell^p \to \ell^p$ l'opérateur de décalage à droite défini par $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$.

Conjecture 2.1 *Supposons $1 < p < \infty$, $p \neq 2$. Soit Ω un espace mesuré et soit $T : L^p(\Omega) \to L^p(\Omega)$ une contraction. Pour tout polynôme complexe P, on a*

$$\|P(T)\|_{L^p(\Omega)\to L^p(\Omega)} \leqslant \|P(S)\|_{\ell^p\to\ell^p}. \tag{1}$$

Il est facile de voir que (1) est vraie pour $p = 1$ et $p = \infty$. De plus, en utilisant la transformation de Fourier, il est clair que pour $p = 2$, (1) est une conséquence de l'inégalité de von Neumann. Notons enfin que, très récemment et après la rédaction de ce chapitre, S. W. Drury [34] a trouvé un contre-exemple pour $p = 4$ en utilisant l'informatique.

Pour les autres valeurs de p, le problème de la validité de (1) pour toute contraction est ouvert. Il est bien connu que l'inégalité (1) est vraie pour toute contraction positive, plus généralement pour tous les opérateurs $T\colon L^p(\Omega) \to L^p(\Omega)$ qui admettent un majorant contractant (i.e. il existe une contraction positive $\tilde{T}$ vérifiant $|T(f)| \leqslant \tilde{T}(|f|)$). Cela vient du fait que ces opérateurs admettent une dilatation isométrique. On renvoie le lecteur à [3], [24], [58], [82] et [89] pour plus d'information sur cette question.

En 1985, V.V. Peller [90] a introduit une version non commutative de la conjecture de Matsaev pour les espaces de Schatten $S^p = S^p(\ell^2)$. Rappelons que les éléments de S^p peuvent être vus comme des matrices infinies indexées par $\mathbb{N} \times \mathbb{N}$. Alors on définit l'application linéaire $\sigma\colon S^p \to S^p$ comme le décalage 'du NO au SE' qui envoie toute matrice

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad \text{sur} \quad \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & a_{00} & a_{01} & \cdots \\ 0 & a_{10} & a_{11} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{2}$$

Soit $S^p(S^p)$ l'espace de toutes les matrices $[a_{ij}]_{i,j\geqslant 0}$ à coefficients a_{ij} dans S^p, qui représentent un élément de l'espace de Schatten $S^p(\ell^2 \otimes_2 \ell^2)$. Le produit tensoriel algébrique $S^p \otimes S^p$ peut être vu comme un sous-espace dense de $S^p(S^p)$ de manière naturelle. Alors l'application donnée par (2) sur $S^p(S^p)$ est une isométrie, qui est l'unique extension de $\sigma \otimes I_{S^p}$ à l'espace $S^p(S^p)$ (voir la section 2 du chapitre 1 pour plus de détails sur ces représentations matricielles). La question de V.V. Peller est la suivante.

Question 2.2 *Supposons $1 < p < \infty$, $p \neq 2$. Soit $T\colon S^p \to S^p$ une contraction sur l'espace de Schatten S^p. A-t-on*

$$\|P(T)\|_{S^p \to S^p} \leqslant \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} \tag{3}$$

pour tout polynôme complexe P ?

Peller a observé que l'inégalité (3) est vraie quand T est une isométrie ou quand l'application $T : S^p \to S^p$ est définie par $T(x) = axb$, où $a : \ell^2 \to \ell^2$ et $b : \ell^2 \to \ell^2$ sont des contractions.

Les espaces de Schatten S^p sont les exemples de base d'espaces L^p non commutatifs. Il est alors naturel d'étendre le problème de Peller à ce contexte plus large. Cela mène à la question suivante.

Question 2.3 *Supposons $1 < p < \infty$, $p \neq 2$. Soit M une algèbre de von Neumann semifinie et soit $L^p(M)$ l'espace L^p non commutatif associé. Soit $T : L^p(M) \to L^p(M)$ une contraction. A-t-on*

$$\|P(T)\|_{L^p(M) \to L^p(M)} \quad \leqslant \quad \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} \tag{4}$$

pour tout polynôme complexe P ?

Comme dans le cas commutatif, il est facile de voir que l'inégalité (4) est vraie quand $p=1$, $p=2$ ou $p=\infty$. Le but principal de ce chapitre est d'exhiber de large classes de contractions sur des espaces L^p non commutatifs qui vérifient l'inégalité (4) pour tout polynôme complexe P. Le théorème suivant réunit certains de nos résultats principaux.

Theorem 2.4 *Supposons $1<p<\infty$. Les applications suivantes vérifient (4) pour tout polynôme complexe P.*

1. *Un multiplicateur de Schur $M_A : S^p \to S^p$ induit par un multiplicateur de Schur $M_A \colon B(\ell^2) \to B(\ell^2)$ contractant associé à une matrice réelle A.*
2. *Un multiplicateur de Fourier $M_t : L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ induit par un multiplicateur de Fourier contractant $M_t : \mathrm{VN}(G) \to \mathrm{VN}(G)$ associé à une fonction à valeurs réelles $t \colon G \to \mathbb{R}$, dans le cas où G est un groupe discret moyennable.*
3. *Un multiplicateur de Fourier $M_t : L^p(\mathrm{VN}(\mathbb{F}_n)) \to L^p(\mathrm{VN}(\mathbb{F}_n))$ induit par un multiplicateur de Fourier unital complètement positif $M_t : \mathrm{VN}(\mathbb{F}_n) \to \mathrm{VN}(\mathbb{F}_n)$ associé à une fonction à valeurs réelles $t : \mathbb{F}_n \to \mathbb{R}$, où $\mathbb{F}_n$ est le groupe libre à n générateurs ($1 \leqslant n \leqslant \infty$).*

Ces résultats reposent sur des théorèmes de dilatation que nous allons maintenant énoncer. De plus, ces théorèmes sont basés sur des constructions dues à É. Ricard [104].

Theorem 2.5 *Soit $M_A : B(\ell^2) \to B(\ell^2)$ un multiplicateur de Schur unital complètement positif associé à une matrice réelle A. Alors il existe une algèbre de von Neumann hyperfinie M munie d'une trace semifinie normale fidèle, un $*$-automorphisme $U : M \to M$ unital préservant la trace et un $*$-monomorphisme $J : B(\ell^2) \to M$ normal unital préservant les traces tels que*

$$(M_A)^k = \mathbb{E} U^k J$$

pour tout entier $k \geqslant 0$, où $\mathbb{E} : M \to B(\ell^2)$ est l'espérance conditionnelle fidèle normale préservant la trace canonique associée à J.

Theorem 2.6 *Soit G un groupe discret. Soit $M_t : \mathrm{VN}(G) \to \mathrm{VN}(G)$ un multiplicateur de Fourier unital complètement positif associé à une fonction à valeurs réelles $t : G \to \mathbb{R}$. Alors il existe une algèbre de von Neumann munie d'une trace finie normale fidèle, un $*$-automorphisme $U : M \to M$ unital préservant la trace et un $*$-monomorphisme $J : \mathrm{VN}(G) \to M$ normal unital préservant les traces tels que*

$$(M_t)^k = \mathbb{E} U^k J$$

pour tout entier $k \geqslant 0$, où $\mathbb{E} : M \to VN(G)$ est l'espérance conditionnelle fidèle normale préservant la trace canonique associée à J. De plus, si G est moyennable ou si $G = \mathbb{F}_n$ $(1 \leqslant n \leqslant \infty)$, l'algèbre de von Neumann M a la propriété QWEP.

Différentes normes sur l'espace des polynômes complexes apparaissent naturellement en examinant la conjecture de Matsaev et le problème de Peller, et il est intéressant d'essayer de les comparer. Si $1 \leqslant p \leqslant \infty$, notons que l'espace de toutes les matrices diagonales de S^p peut être identifié avec ℓ^p. Avec ce point de vue, l'opérateur de décalage $S : \ell^p \to \ell^p$ coïncide avec la restriction de $\sigma : S^p \to S^p$ aux matrices diagonales. Ceci implique immédiatement que

$$\|P(S)\|_{\ell^p \to \ell^p} \leqslant \|P(\sigma)\|_{S^p \to S^p} \leqslant \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)}$$

pour tout polynôme complexe P. On montrera le résultat suivant, qui contredit une conjecture de Peller [90, Conjecture 2].

Theorem 2.7 *Supposons $1 < p < \infty$, $p \neq 2$. Alors il existe un polynôme complexe P tel que*

$$\|P(S)\|_{\ell^p \to \ell^p} < \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)}.$$

Pour compléter cette investigation, on prouvera que

$$\|P(\sigma)\|_{S^p \to S^p} = \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} = \|P(S) \otimes Id_{S^p}\|_{\ell^p(S^p) \to \ell^p(S^p)} \qquad (5)$$

pour tout P (la première de ces inégalités étant due à É. Ricard).

Ce chapitre est organisé de la manière suivante. Dans la section 2, on fixe certaines notations, on donne des informations sur la notion clé d'application complètement bornée sur un espace L^p non commutatif, on prouve la seconde inégalité de (5) et on donne certains résultats préliminaires. Dans la section 3, on montre que certains multiplicateurs de Fourier sur $L^p(\mathbb{R})$ et $\ell^p_{\mathbb{Z}}$ sont bornés mais pas complètement bornés et on prouve le théorème 2.7 et la première égalité de (5). La section suivante 4 est consacrée aux classes de contractions qui vérifient l'inégalité de Matsaev non commutative (4) pour tout polynôme complexe P. En particulier, on prouve les théorèmes 2.5 et 2.6. Dans la section 5, on considère un analogue naturel de la question 2.3 pour les C_0-semigroupes de contractions. Finalement dans la dernière section 6, on exhibe des polynômes P qui vérifient toujours (4) pour toute contraction T.

2.2 Chapitre 2

Soit (Ω, μ) un espace mesuré et soit $1 < p < \infty$. Pour tout opérateur borné $T\colon L^p(\Omega) \to L^p(\Omega)$, considérons la 'fonction carrée'

$$\|x\|_{T,1} = \left\| \Big(\sum_{k=1}^{+\infty} k \big| T^k(x) - T^{k-1}(x) \big|^2 \Big)^{\frac{1}{2}} \right\|_{L^p}, \tag{6}$$

définie pour tout $x \in L^p(\Omega)$. De telles quantités apparaissent fréquemment dans l'analyse des opérateurs sur des espaces L^p. Elles remontent au moins à [109], où elles sont utilisées en lien avec les fonctions carrées de martingales pour étudier des semigroupes de diffusion et leurs contreparties discrètes. Des fonctions carrées similaires pour les semigroupes continus ont joué un rôle clé dans le récent développement du calcul fonctionnel H^∞ et de ses applications (voir en particulier le papier [52], l'article de synthèse [67] et leurs références).

Soit $(\varepsilon_k)_{k \geqslant 1}$ une suite de variables de Rademacher indépendantes sur un certain espace probabilisé Ω_0. Soit X un espace de Banach. On définit $\mathrm{Rad}(X) \subset L^2(\Omega_0, X)$ comme étant l'adhérence du sous-espace engendré par $\{\varepsilon_k \otimes x \,:\, k \geqslant 1,\ x \in X\}$ dans l'espace de Bochner $L^2(\Omega_0, X)$. Les deux définitions suivantes sont fondamentales pour ce chapitre. La première est celle de la R-bornitude.

Definition 2.8 *On dit qu'un ensemble $F \subset B(X)$ est R-borné s'il existe une constante $C \geqslant 0$ telle que pour toute familles finies $T_1, \ldots, T_n$ de F et $x_1, \ldots, x_n$ de X, on ait*

$$\left\| \sum_{k=1}^{n} \varepsilon_k \otimes T_k(x_k) \right\|_{\mathrm{Rad}(X)} \leqslant C \left\| \sum_{k=1}^{n} \varepsilon_k \otimes x_k \right\|_{\mathrm{Rad}(X)}.$$

La seconde est celle d'un analogue discret de la propriété d'analyticité pour les semigroupes continus.

Definition 2.9 *On dit qu'un opérateur $T \in B(X)$ est un opérateur de Ritt si les deux ensembles*

$$\{T^n \,:\, n \geqslant 0\} \qquad \text{et} \qquad \Big\{ n(T^n - T^{n-1}) \,:\, n \geqslant 1 \Big\} \tag{7}$$

sont bornés. De même, on dit que T est un opérateur R-Ritt si les deux ensembles de (7) sont R-bornés.

L'article [69] contient une preuve du fait que si T est à la fois une contraction positive et un opérateur de Ritt, alors il vérifie une estimation uniforme $\|x\|_{T,1} \lesssim \|x\|_{L^p}$ pour $x \in L^p(\Omega)$. Cette estimation et d'autres du même genre mènent à des inégalités maximales fortes pour cette classe d'opérateurs (voir aussi [70]). De plus, dans l'article [68], on trouve une étude des opérateurs $T\colon L^p(\Omega) \to L^p(\Omega)$ tels que T et son opérateur adjoint $T^*\colon L^{p^*}(\Omega) \to L^{p^*}(\Omega)$ admettent tous les deux des estimations uniformes

$$\|x\|_{T,1} \lesssim \|x\|_{L^p} \qquad \text{et} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}} \tag{8}$$

pour $x \in L^p(\Omega)$ et $y \in L^{p^*}(\Omega)$. (Ici $p^* = \frac{p}{p-1}$ est le nombre conjugué de p.) Il est démontré que (8) implique que T est un opérateur R-Ritt et que (8) est équivalent à ce que T ait un calcul fonctionnel H^∞ borné par rapport à un domaine de Stolz du disque unité de sommet 1.

Le présent chapitre est une suite de ces investigations. Notre principal résultat est le théorème suivant. Il donne une caractérisation de (8) en termes de dilatations. On montre que (8) est vraie si et seulement si T est R-Ritt et s'il admet une dilatation grossière (loose dilation) i.e. il existe un autre espace mesuré $(\widetilde{\Omega}, \widetilde{\mu})$, deux applications bornées $J\colon L^p(\Omega) \to L^p(\widetilde{\Omega})$ et $Q\colon L^p(\widetilde{\Omega}) \to L^p(\Omega)$, ainsi qu'un isomorphisme $U\colon L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ tels que l'ensemble $\{U^n \,:\, n \in \mathbb{Z}\}$ soit borné et

$$T^n = QU^nJ, \qquad n \geqslant 0.$$

Theorem 2.10 *Supposons $1 < p < \infty$. Soit $T\colon L^p(\Omega) \to L^p(\Omega)$ un opérateur de Ritt. Les assertions suivantes sont équivalentes.*

(i) L'opérateur T et son adjoint $T^*\colon L^{p^*}(\Omega) \to L^{p^*}(\Omega)$ admettent tous les deux des estimations uniformes

$$\|x\|_{T,1} \lesssim \|x\|_{L^p} \qquad \text{and} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}}$$

pour $x \in L^p(\Omega)$ et $y \in L^{p^*}(\Omega)$.

(ii) L'opérateur T est R-Ritt et admet une dilatation grossière.

Ce résultat sera établi à la section 4. Il devrait être vu comme un analogue discret du résultat principal de [43].

Dans la section 3, on considère des variantes de (6) de la manière suivante. Supposons que $T\colon L^p(\Omega) \to L^p(\Omega)$ soit un opérateur de Ritt. Alors $I - T$ est un opérateur sectoriel et on peut définir ses puissances fractionnaires $(I-T)^\alpha$ pour tout $\alpha > 0$. Alors on considère

$$\|x\|_{T,\alpha} = \left\| \Big(\sum_{k=1}^{+\infty} k^{2\alpha-1} \big|T^{k-1}(I-T)^\alpha x\big|^2 \Big)^{\frac{1}{2}} \right\|_{L^p} \tag{9}$$

pour tout $x \in L^p(\Omega)$. Notre second résultat principal est le théorème suivant qui affirme que quand T est un opérateur R-Ritt, alors les fonctions carrées $\|\cdot\|_{T,\alpha}$ sont deux à deux équivalentes.

Theorem 2.11 *Supposons $1 < p < \infty$. Soit $T\colon L^p(\Omega) \to L^p(\Omega)$ un opérateur R-Ritt. Alors pour tous $\alpha, \beta > 0$, on a une équivalence*

$$\|x\|_{T,\alpha} \approx \|x\|_{T,\beta}, \qquad x \in L^p(\Omega).$$

Ce résultat devrait être vu comme un analogue discret de [66, Théorème 1.1]. On le prouve ici comme une étape clé de notre caractérisation de (8) en termes de dilatations.

Ce chapitre est organisé de la manière suivante. La section 2 contient principalement des résultats préliminaires. La section suivante 3 contient une preuve du théorème 2.11. Dans la section 4, on prouve le théorème 2.10. La section 5 est consacrée à des compléments sur les opérateurs définis sur un espace L^p et leurs propriétés de calcul fonctionnel, en lien avec les applications p-complètement bornées. Finalement, la section 6 contient des généralisations aux opérateurs $T\colon X \to X$ sur des espaces de Banach généraux X. On donne une attention particulière aux espaces L^p non commutatifs, dans l'esprit de [52].

2.3 Chapitre 3

Soit M une algèbre de von Neumann semifinie munie d'une trace semifinie fidèle normale. Pour tout $1 \leqslant p < \infty$, on note $L^p(M)$ l'espace L^p associé (non commutatif). Soit T un opérateur borné sur $L^p(M)$. Considérons la 'fonction carrée' suivante

$$\|x\|_{T,1} = \inf\left\{\left\|\Big(\sum_{k=1}^{+\infty}|u_k|^2\Big)^{\frac{1}{2}}\right\|_{L^p} + \left\|\Big(\sum_{k=1}^{+\infty}|v_k^*|^2\Big)^{\frac{1}{2}}\right\|_{L^p} : \ u_k + v_k = k^{\frac{1}{2}}\big(T^k(x) - T^{k-1}(x)\big)\ \forall\ k \geqslant 1\right\} \tag{10}$$

si $1 < p \leqslant 2$ et

$$\|x\|_{T,1} = \max\left\{\left\|\Big(\sum_{k=1}^{+\infty} k\left|T^k(x) - T^{k-1}(x)\right|^2\Big)^{\frac{1}{2}}\right\|_{L^p}, \left\|\Big(\sum_{k=1}^{+\infty} k\left|\big(T^k(x) - T^{k-1}(x)\big)^*\right|^2\Big)^{\frac{1}{2}}\right\|_{L^p}\right\} \tag{11}$$

si $2 \leqslant p < \infty$, définie pour tout $x \in L^p(M)$. De telles quantités ont été introduites dans [68] et étudiées dans cet article et dans le chapitre 2. Notons en effet que dans le cas commutatif, les fonctions carrées (10) et (11) se réduisent à (6). Pour tout $\gamma \in \left]0, \frac{\pi}{2}\right[$, soit B_γ l'intérieur de l'enveloppe convexe de 1 et du disque unité $D(0, \sin\gamma)$. Supposons $1 < p < \infty$. Soit T un opérateur de Ritt avec $\operatorname{Ran}(I - T)$ dense dans $L^p(M)$ admettant un calcul fonctionnel $H^\infty(B_\gamma)$ borné pour un certain $\gamma \in \left]0, \frac{\pi}{2}\right[$, i.e. il existe un angle $\gamma \in \left]0, \frac{\pi}{2}\right[$ et une constante positive K telle que $\|\varphi(T)\|_{L^p(M)\to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$ pour tout polynôme complexe φ. Un résultat de [68] dit essentiellement que

$$\|x\|_{L^p(M)} \approx \|x\|_{T,1}, \qquad x \in L^p(M) \tag{12}$$

(voir aussi la remarque 6.4 du chapitre 2). Maintenant, considérons les 'fonctions carrées colonne et ligne' suivantes

$$\|x\|_{T,c,1} = \left\|\Big(\sum_{k=1}^{+\infty} k\left|T^k(x) - T^{k-1}(x)\right|^2\Big)^{\frac{1}{2}}\right\|_{L^p} \text{ et } \|x\|_{T,r,1} = \left\|\Big(\sum_{k=1}^{+\infty} k\left|\big(T^k(x) - T^{k-1}(x)\big)^*\right|^2\Big)^{\frac{1}{2}}\right\|_{L^p} \tag{13}$$

définies pour tout $x \in L^p(M)$. Supposons $1 < p < 2$. Dans ce contexte, si $x \in L^p(M)$, il est naturel de chercher des conditions suffisantes pour trouver une décomposition $x = x_1 + x_2$ telle que $\|x_1\|_{T,c,1}$ et

$\|x_2\|_{T,r,1}$ soient finies. Le premier résultat principal de ce chapitre est le théorème suivant. Il renforce l'équivalence précédente (12) dans le cas où T admet en fait un calcul fonctionnel $H^\infty(B_\gamma)$ *complètement* borné, i.e. il existe une constante positive K telle que $\|\varphi(T)\|_{cb,L^p(M)\to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$ pour tout polynôme complexe φ.

Theorem 2.12 *Supposons $1<p<2$. Soit T un opérateur de Ritt sur $L^p(M)$ avec $\mathrm{Ran}(I-T)$ dense dans $L^p(M)$. Supposons que T admette un calcul fonctionnel $H^\infty(B_\gamma)$ complètement borné pour un certain $\gamma \in \left]0, \frac{\pi}{2}\right[$. Alors on a*

$$\|x\|_{L^p(M)} \approx \inf\Big\{\|x_1\|_{T,c,1} + \|x_2\|_{T,r,1} \ : \ x = x_1 + x_2\Big\}, \qquad x \in L^p(M).$$

Dans ce contexte, il est naturel de comparer les deux quantités de (13). Le second résultat principal de ce chapitre est le théorème suivant. Il affirme qu'en général, 'les fonctions carrées colonne et ligne' ne sont pas équivalentes.

Theorem 2.13 *Supposons $1<p<\infty$, $p \neq 2$. Alors il existe un opérateur de Ritt T sur l'espace de Schatten S^p, avec $\mathrm{Ran}(I-T)$ dense dans S^p, admettant un calcul fonctionnel $H^\infty(B_\gamma)$ complètement borné pour un certain $\gamma \in \left]0, \frac{\pi}{2}\right[$ tel que*

$$\sup\left\{\frac{\|x\|_{T,c,1}}{\|x\|_{T,r,1}} \ : \ x \in S^p\right\} = \infty \ \text{ if } 2<p<\infty \ \text{ et } \sup\left\{\frac{\|x\|_{T,r,1}}{\|x\|_{T,c,1}} \ : \ x \in S^p\right\} = \infty \ \text{ if } 1<p<2. \tag{14}$$

De plus, le même résultat est vrai avec les rôles de $\|\cdot\|_{T,c,1}$ et $\|\cdot\|_{T,r,1}$ échangés.

Pour un opérateur de Ritt admettant un calcul fonctionnel $H^\infty(B_\gamma)$ complètement borné, il semble aussi intéressant, au vu de l'équivalence (12), de comparer ces deux quantités avec la norme usuelle $\|\cdot\|_{L^p(M)}$. Si T est un opérateur de Ritt avec $\mathrm{Ran}(I-T)$ dense dans $L^p(M)$ admettant un calcul fonctionnel $H^\infty(B_\gamma)$ borné pour un certain $\gamma \in \left]0, \frac{\pi}{2}\right[$, l'équivalence (12) implique que

$$\|x\|_{L^p(M)} \lesssim \|x\|_{T,c,1} \qquad \text{et} \qquad \|x\|_{L^p(M)} \lesssim \|x\|_{T,r,1}$$

si $1<p\leqslant 2$ et

$$\|x\|_{T,c,1} \lesssim \|x\|_{L^p(M)} \qquad \text{et} \qquad \|x\|_{T,r,1} \lesssim \|x\|_{L^p(M)}$$

si $2 \leqslant p < \infty$, pour tout $x \in L^p(M)$. Le dernier résultat principal de ce chapitre est que, excepté si $p=2$, ces estimations ne peuvent pas être renversées :

Theorem 2.14 *Supposons $2<p<\infty$ (resp. $1<p<2$). Il existe un opérateur de Ritt T sur l'espace de Schatten S^p, avec $\mathrm{Ran}(I-T)$ dense dans S^p, admettant un calcul fonctionnel $H^\infty(B_\gamma)$ complètement borné avec $\gamma \in \left]0, \frac{\pi}{2}\right[$ tel que*

$$\sup\left\{\frac{\|x\|_{S^p}}{\|x\|_{T,c,1}} \ : \ x \in S^p\right\} = \infty \quad \left(\textit{resp. } \sup\left\{\frac{\|x\|_{T,c,1}}{\|x\|_{S^p}} \ : \ x \in S^p\right\} = \infty\right).$$

De plus, le même résultat est vrai avec $\|\cdot\|_{T,c,1}$ *remplacé par* $\|\cdot\|_{T,r,1}$.

Ce chapitre est organisé de la manière suivante. La section 2 donne une brève présentation des espaces L^p non commutatifs et des opérateurs de Ritt et on introduit les notions d'opérateurs Col-Ritt et Row-Ritt et de calcul fonctionnel $H^\infty(B_\gamma)$ complètement borné qui sont utiles pour ce chapitre. La section 3 suivante contient principalement des résultats préliminaires concernant les opérateurs Col-Ritt et Row-Ritt. La section 4 est consacrée à prouver les théorèmes 2.13 and 2.14. Dans la section 5, on présente une preuve du théorème 2.12. On termine cette section en donnant certains exemples naturels auxquels ce résultat peut être appliqué.

2.4 Chapitre 4

L'algèbre de Fourier $A(G)$ d'un groupe localement compact G fut introduite par P. Eymard dans [39]. L'algèbre $A(G)$ est le prédual de l'algèbre de von Neumann $VN(G)$ du groupe G. Si G est abélien de groupe dual $\widehat{G}$, alors la transformation de Fourier induit un isomorphisme isométrique de $L^1(\widehat{G})$ sur $A(G)$. Dans [41], A. Figà-Talamanca a montré que, si G est abélien, le prédual naturel de l'espace de Banach des multiplicateurs de Fourier bornés sur $L^p(G)$ est isométriquement isomorphe à un espace $A_p(G)$ de fonctions continues sur G. De plus, on a $A_2(G) = A(G)$ isométriquement. Dans [39] et [47], C. Herz a prouvé que l'espace $A_p(G)$ est une algèbre de Banach pour le produit usuel des fonctions (voir aussi [91]). Donc $A_p(G)$ est un analogue 'L^p' de l'algèbre de Fourier $A(G)$. Ces algèbres sont appelées algèbres de Figà-Talamanca-Herz. Dans [105], V. Runde a introduit un espace d'opérateurs analogue $OA_p(G)$ de l'algèbre $A_p(G)$. L'espace de Banach sous-jacent à $OA_p(G)$ est différent de l'espace de Banach $A_p(G)$. De plus, il est possible de montrer (en utilisant une variante convenable de [64, Théorème 5.6.1]) que $OA_p(G)$ est le prédual naturel de l'espace d'opérateurs des multiplicateurs de Fourier complètement bornés. On renvoie à [28], [29], [63] et [106] pour d'autres espaces d'opérateurs analogues de $A_p(G)$.

Le but de ce chapitre est d'introduire des analogues non commutatifs de ces algèbres dans le contexte des multiplicateurs de Schur complètement bornés sur les espaces de Schatten S^p. Rappelons qu'une application $T\colon S^p \to S^p$ est complètement bornée si $Id_{S^p} \otimes T$ est borné sur $S^p(S^p)$. Si $1 \leqslant p < \infty$, l'espace d'opérateurs $CB(S^p)$ des applications complètement bornées de S^p dans lui-même est naturellement un espace d'opérateurs dual. En effet, on a un isomorphisme isométrique $CB(S^p) = \big(S^p \widehat{\otimes} S^{p^*}\big)^*$ où $\widehat{\otimes}$ désigne le produit tensoriel projectif d'espace d'opérateurs. De plus, on prouvera que le sous-espace $\mathfrak{M}_{p,cb}$ des multiplicateurs de Schur complètement bornés est un sous-ensemble commutatif maximal de $CB(S^p)$. Par conséquent, le sous-espace $\mathfrak{M}_{p,cb}$ est préfaiblement fermé dans $CB(S^p)$. Donc $\mathfrak{M}_{p,cb}$ est naturellement un espace d'opérateurs dual avec $\mathfrak{M}_{p,cb} = \big(S^p \widehat{\otimes} S^{p^*} / (\mathfrak{M}_{p,cb})_\perp\big)^*$. Si on désigne par $\psi_p\colon S^p \widehat{\otimes} S^{p^*} \to S^1$ l'application $A \otimes B \mapsto A * B$, où $*$ désigne le produit de Schur, on montrera que $(\mathfrak{M}_{p,cb})_\perp = \operatorname{Ker} \psi_p$. Maintenant, on définit l'espace d'opérateurs $\mathfrak{R}_{p,cb}$ comme l'espace $\operatorname{Im} \psi_p$ muni de la structure d'espace d'opérateurs de $S^p \widehat{\otimes} S^{p^*} / \operatorname{Ker} \psi_p$. On a complètement isométriquement $(\mathfrak{R}_{p,cb})^* = \mathfrak{M}_{p,cb}$. De plus, par définition, on a une inclusion complètement contractante $\mathfrak{R}_{p,cb} \subset S^1$.

Rappelons que les éléments de S^1 peuvent être vus comme des matrices infinies. Notre principal résultat est le théorème suivant.

Theorem 2.15 *Supposons $1 \leqslant p < \infty$. Le prédual $\mathfrak{R}_{p,cb}$ de l'espace d'opérateurs $\mathfrak{M}_{p,cb}$ muni du produit matriciel usuel ou du produit de Schur est une algèbre de Banach complètement contractante.*

Dans [85] et [111], S. K. Parott et R. S. Strichartz ont montré que si $1 \leqslant p \leqslant \infty$, $p \neq 2$ tout multiplicateur de Fourier isométrique sur $L^p(G)$ est un multiple scalaire d'un opérateur induit par une translation. Dans [41], A. Figà-Talamanca a montré que l'espace des multiplicateurs de Fourier bornés est l'adhérence pour la topologie faible d'opérateurs de l'espace vectoriel engendré par ces opérateurs. On donne des analogues non commutatifs de ces deux résultats.

Theorem 2.16

1. *Supposons $1 \leqslant p \leqslant \infty$, $p \neq 2$. Alors tout multiplicateur de Schur isométrique sur S^p est défini par une matrice $[a_i b_j]$ avec $a_i, b_j \in \mathbb{T}$.*
2. *Supposons $1 \leqslant p < \infty$. L'espace $\mathfrak{M}_p$ des multiplicateurs de Schur bornés sur S^p est l'adhérence de l'espace vectoriel engendré par les multiplicateurs de Schur isométriques pour la topologie d'opérateurs faible.*

Ce chapitre est organisé de la manière suivante. Dans la section 2, on fixe certaines notations et on montre que les préduaux naturels de $\mathfrak{M}_p$ et $\mathfrak{M}_{p,cb}$ admettent des réalisations concrètes comme espace de matrices. On donne aussi des propriétés élémentaires de ces espaces. Dans la section 3, on montre que le prédual naturel $\mathfrak{R}_{p,cb}$ de l'espace d'opérateurs $\mathfrak{R}_{p,cb}$ des multiplicateurs de Schur complètement bornés, muni du produit matriciel, est une algèbre de Banach complètement contractante. Dans la section 4, on se tourne vers le produit de Schur. On observe que le prédual naturel $\mathfrak{R}_p$ de l'espace de Banach $\mathfrak{M}_p$ des multiplicateurs de Schur bornés est une algèbre de Banach pour le produit de Schur. De plus, on montre que l'espace $\mathfrak{R}_{p,cb}$ muni du produit de Schur est une algèbre de Banach complètement contractante. Dans la dernière section 5, on détermine les multiplicateurs de Schur isométriques sur S^p et on prouve que l'espace $\mathfrak{M}_p$ est l'adhérence pour la topologie faible d'opérateurs de l'espace vectoriel engendré par les multiplicateurs isométriques.

Chapitre I

On Matsaev's conjecture for contractions on noncommutative L^p-spaces

1 Introduction

To estimate the norms of functions of operators is an essential task in Operator Theory. In this subject, V. V. Matsaev stated the following conjecture in 1971, see [81]. For any $1 \leqslant p \leqslant \infty$, let $\ell^p \xrightarrow{S} \ell^p$ denote the right shift operator defined by $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$.

Conjecture 1.1 *Suppose $1 < p < \infty$, $p \neq 2$. Let Ω be a measure space and let $L^p(\Omega) \xrightarrow{T} L^p(\Omega)$ be a contraction. For any complex polynomial P, we have*

$$\|P(T)\|_{L^p(\Omega) \to L^p(\Omega)} \leqslant \|P(S)\|_{\ell^p \to \ell^p}. \tag{1.1}$$

It is easy to see that (1.1) holds true for $p = 1$ and $p = \infty$. Moreover, by using the Fourier transform, it is clear that for $p = 2$, (1.1) is a consequence of von Neumann's inequality. Finally, very recently and after the writing of this chapter, S. W. Drury [34] found a counterexample in the case $p = 4$ by using computer.

For all other values of p, the validity of (1.1) for any contraction is open. It is well-known that (1.1) holds true for any positive contraction, more generally for all operators $L^p(\Omega) \xrightarrow{T} L^p(\Omega)$ which admit a contractive majorant (i.e. there exists a positive contraction $\tilde{T}$ satisfying $|T(f)| \leqslant \tilde{T}(|f|)$). This follows from the fact that these operators admit an isometric dilation. We refer the reader to [3], [24], [58], [82] and [89] for information and historical background on this question.

In 1985, V.V. Peller [90] introduced a noncommutative version of Matsaev's conjecture for Schatten spaces $S^p = S^p(\ell^2)$. Recall that elements of S^p can be regarded as infinite matrices indexed by $\mathbb{N} \times \mathbb{N}$.

Thus we define the linear map $S^p \xrightarrow{\sigma} S^p$ as the shift 'from NW to SE' which maps any matrix

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & a_{00} & a_{01} & \cdots \\ 0 & a_{10} & a_{11} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{1.2}$$

Let $S^p(S^p)$ be the space of all matrices $[a_{ij}]_{i,j\geqslant 0}$ with entries a_{ij} in S^p, which represent an element of the bigger Schatten space $S^p(\ell^2 \otimes_2 \ell^2)$. The algebraic tensor product $S^p \otimes S^p$ can be regarded as a dense subspace of $S^p(S^p)$ in a natural way. Then the mapping on $S^p(S^p)$ given by (1.2) is an isometry, which is the unique extension of $\sigma \otimes I_{S^p}$ to the space $S^p(S^p)$. (See Section 2 below for more details on these matricial representations.) Peller's question is as follows.

Question 1.2 *Suppose $1 < p < \infty$, $p \neq 2$. Let $S^p \xrightarrow{T} S^p$ be a contraction on the Schatten space S^p. Do we have*

$$\|P(T)\|_{S^p \to S^p} \leqslant \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} \tag{1.3}$$

for any complex polynomial P?

Peller observed that (1.3) holds true when T is an isometry or when $S^p \xrightarrow{T} S^p$ is defined by $T(x) = axb$, where $\ell^2 \xrightarrow{a} \ell^2$ and $\ell^2 \xrightarrow{b} \ell^2$ are contractions.

The Schatten spaces S^p are basic examples of noncommutative L^p-spaces. It is then natural to extend Peller's problem to this wider context. This leads to the following question.

Question 1.3 *Suppose $1 < p < \infty$, $p \neq 2$. Let M be a semifinite von Neumann algebra and let $L^p(M)$ be the associated noncommutative L^p-space. Let $L^p(M) \xrightarrow{T} L^p(M)$ be a contraction. Do we have*

$$\|P(T)\|_{L^p(M) \to L^p(M)} \quad \leqslant \quad \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} \tag{1.4}$$

for any complex polynomial P?

As in the commutative case, it is easy to see that (1.4) holds true when $p = 1$, $p = 2$ or $p = \infty$. The main purpose of this article is to exhibit large classes of contractions on noncommutative L^p-spaces which satisfy inequality (1.4) for any complex polynomial P. The next theorem gathers some of our main results.

Theorem 1.4 *Suppose $1 < p < \infty$. The following maps satisfy (1.4) for any complex polynomial P.*

1. *A Schur multiplier $S^p \xrightarrow{M_A} S^p$ induced by a contractive Schur multiplier $B(\ell^2) \xrightarrow{M_A} B(\ell^2)$ associated with a real-valued matrix A.*

2. *A Fourier multiplier* $L^p(\mathrm{VN}(G)) \xrightarrow{M_t} L^p(\mathrm{VN}(G))$ *induced by a contractive Fourier multiplier* $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ *associated with a real valued function* $G \xrightarrow{t} \mathbb{R}$*, in the case where* G *is an amenable discrete group* G*.*
3. *A Fourier multiplier* $L^p(\mathrm{VN}(\mathbb{F}_n)) \xrightarrow{M_t} L^p(\mathrm{VN}(\mathbb{F}_n))$ *induced by a unital completely positive Fourier multiplier* $\mathrm{VN}(\mathbb{F}_n) \xrightarrow{M_t} \mathrm{VN}(\mathbb{F}_n)$ *associated with a real valued function* $\mathbb{F}_n \xrightarrow{t} \mathbb{R}$*, where* $\mathbb{F}_n$ *is the free group with* n *generators* ($1 \leqslant n \leqslant \infty$)*.*

The proof of these results will use dilation theorems that we now state. Moreover, these theorems rely on constructions dues to É. Ricard [104].

Theorem 1.5 *Let* $B(\ell^2) \xrightarrow{M_A} B(\ell^2)$ *be a unital completely positive Schur multiplier with a real-valued matrix* A*. Then there exists a hyperfinite von Neumann algebra* M *equipped with a semifinite normal faithful trace, a unital trace preserving* $*$*-automorphism* $M \xrightarrow{U} M$*, a unital trace preserving one-to-one normal* $*$*-homomorphism* $B(\ell^2) \xrightarrow{J} M$ *such that*

$$(M_A)^k = \mathbb{E}U^k J$$

for any integer $k \geqslant 0$*, where* $M \xrightarrow{\mathbb{E}} B(\ell^2)$ *is the canonical faithful normal trace preserving conditional expectation associated with* J*.*

Theorem 1.6 *Let* G *be a discrete group. Let* $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ *be a unital completely positive Fourier multiplier associated with a real valued function* $G \xrightarrow{t} \mathbb{R}$*. Then there exists a von Neumann algebra* M *equipped with a faithful finite normal trace, a unital trace preserving* $*$*-automorphism* $M \xrightarrow{U} M$*, a unital normal trace preserving one-to-one* $*$*-homomorphism* $\mathrm{VN}(G) \xrightarrow{J} M$ *such that,*

$$(M_t)^k = \mathbb{E}U^k J$$

for any integer $k \geqslant 0$*, where* $M \xrightarrow{\mathbb{E}} VN(G)$ *is the canonical faithful normal trace preserving conditional expectation associated with* J*. Moreover, if* G *is amenable or if* $G = \mathbb{F}_n$ ($1 \leqslant n \leqslant \infty$)*, the von Neumann algebra* M *has the quotient weak expectation property.*

Various norms on the space of complex polynomials arise from Matsaev's conjecture and Peller's problem, and it is interesting to try to compare them. If $1 \leqslant p \leqslant \infty$, note that the space of all diagonal matrices in S^p can be identified with ℓ^p. In this regard, the shift operator $\ell^p \xrightarrow{S} \ell^p$ coincides with the restriction of $S^p \xrightarrow{\sigma} S^p$ to diagonal matrices. This readily implies that

$$\|P(S)\|_{\ell^p \to \ell^p} \leqslant \|P(\sigma)\|_{S^p \to S^p} \leqslant \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)}$$

for any complex polynomial P. We will show the following result, which disproves a conjecture due to Peller [90, Conjecture 2].

Theorem 1.7 *Suppose $1 < p < \infty$, $p \neq 2$. Then there exists a complex polynomial P such that*

$$\|P(S)\|_{\ell^p \to \ell^p} < \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)}.$$

To complete this investigation, we will also show that

$$\|P(\sigma)\|_{S^p \to S^p} = \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} = \|P(S) \otimes Id_{S^p}\|_{\ell^p(S^p) \to \ell^p(S^p)} \tag{1.5}$$

for any P (the first of these equalities being due to É. Ricard).

The paper is organized as follows. In §2, we fix some notations, we give some background on the key notion of completely bounded maps on noncommutative L^p-spaces, we prove the second equality of (1.5) and we give some preliminary results. In §3, we show that some Fourier multipliers on $L^p(\mathbb{R})$ and $\ell^p_{\mathbb{Z}}$ are bounded but not completely bounded and we prove Theorem 1.7 and the first equality of (1.5). §4 is devoted to classes of contractions which satisfy noncommutative Matsaev's inequality (1.4) for any complex polynomial P. In particular we prove Theorems 1.5 and 1.6. In §5, we consider a natural analog of Question 1.3 for C_0-semigroups of contractions. Finally in §6, we exhibit some polynomials P which always satisfy (1.4) for any contraction T.

2 Preliminaries

Let us recall some basic notations. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\delta_{i,j}$ the symbol of Kronecker.

If I is an index set and if E is a vector space, we write $\mathbb{M}_I$ for the space of the $I \times I$ matrices with entries in $\mathbb{C}$ and $\mathbb{M}_I(E)$ for the space of the $I \times I$ matrices with entries in E. If K is another index set, we have an isomorphism $\mathbb{M}_I(\mathbb{M}_K) = \mathbb{M}_{I \times K}$.

Let M be a von Neumann algebra equipped with a semifinite normal faithful trace τ. For $1 \leqslant p < \infty$ the noncommutative L^p-space $L^p(M)$ is defined as follows. If S^+ is the set of all positive $x \in M$ such that $\tau(x) < \infty$ and S is its linear span, then $L^p(M)$ is the completion of S with respect to the norm $\|x\|_{L^p(M)} = \tau\big(|x|^p\big)^{\frac{1}{p}}$. One sets $L^\infty(M) = M$. We refer to [103], and the references therein, for more information on these spaces.

Let $1 \leqslant p < \infty$. If I is an index set and if we equip the space $B(\ell^2_I)$ with the operator norm and the canonical trace Tr, the space $L^p\big(B(\ell^2_I)\big)$ identifies to the Schatten-von Neumann class S^p_I. The space S^p_I is the space of those compact operators x from ℓ^2_I into ℓ^2_I such that $\|x\|_{S^p_I} = \big(\mathrm{Tr}\,(x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$. The space S^∞_I of compact operators from ℓ^2_I into ℓ^2_I is equipped with the operator norm. For $I = \mathbb{N}$, we simplify the notations, we let S^p for $S^p_{\mathbb{N}}$. Elements of S^p_I are regarded as matrices $A = [a_{ij}]_{i,j \in I}$ of $\mathbb{M}_I$. The space $S^p_I(S^p_K)$ is the space of those compact operators x from $\ell^2_I \otimes_2 \ell^2_K$ into $\ell^2_I \otimes_2 \ell^2_K$ such that $\|x\|_{S^p_I(S^p_K)} = \big((\mathrm{Tr} \otimes \mathrm{Tr}\,)(x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$. Elements of $S^p_I(S^p_K)$ are regarded as matrices of $\mathbb{M}_I(\mathbb{M}_K)$.

Let M be a von Neumann algebra equipped with a semifinite normal faithful trace τ. If the von Neumann algebra $B(\ell^2_I)\overline{\otimes} M$ is equipped with the semifinite normal faithful trace $\mathrm{Tr}\,\otimes\tau$, the space

$L^p(B(\ell^2_I)\overline{\otimes}M)$ identifies to a space $S^p_I(L^p(M))$ of matrices of $\mathbb{M}_I(L^p(M))$. Moreover, under this identification, the algebraic tensor product $S^p_I \otimes L^p(M)$ is dense in $S^p_I(L^p(M))$.

Let N be another von Neumann algebra equipped with a semifinite normal faithful trace. If $1 \leqslant p \leqslant \infty$, we say that a linear map $L^p(M) \xrightarrow{T} L^p(N)$ is completely bounded if $Id_{S^p} \otimes T$ extends to a bounded operator $S^p(L^p(M)) \xrightarrow{Id_{S^p}\otimes T} S^p(L^p(N))$. In this case, the completely bounded norm $\|T\|_{cb,L^p(M)\to L^p(N)}$ is defined by

$$\|T\|_{cb,L^p(M)\to L^p(N)} = \|Id_{S^p} \otimes T\|_{S^p(L^p(M))\to S^p(L^p(N))}. \tag{2.1}$$

If Ω is a measure space, the space $S^p(L^p(\Omega))$ is isometric to the L^p-space $L^p(\Omega, S^p)$ of S^p-valued functions in Bochner's sense. Thus, if $L^p(\Omega) \xrightarrow{T} L^p(\Omega)$ is a linear map, we have

$$\|T\|_{cb,L^p(\Omega)\to L^p(\Omega)} = \|T \otimes Id_{S^p}\|_{L^p(\Omega,S^p)\to L^p(\Omega,S^p)}. \tag{2.2}$$

The notion of completely bounded map and the completely bounded norm defined in (2.1) are the same that these defined in operator space theory, see [37], [99] and [101].

Now, we let:

Definition 2.1 *Let M be a von Neumann algebra equipped with a faithful semifinite normal trace and $1 \leqslant p \leqslant \infty$. Let $L^p(M) \xrightarrow{T} L^p(M)$ be a contraction. We say that T satisfies the noncommutative Matsaev's property if (1.4) holds for any complex polynomial P.*

We denote by $\ell^p \xrightarrow{S} \ell^p$ the right shift on ℓ^p. We use the same notation for the right shift on $\ell^p_{\mathbb{Z}}$. We denote by S_- the left shift on ℓ^p defined by $S_-(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$. Suppose $1 \leqslant p \leqslant \infty$. Let X be a Banach space. For any complex polynomial P, we define $\|P\|_{p,X}$ by

$$\|P\|_{p,X} = \|P(S) \otimes Id_X\|_{\ell^p(X)\to\ell^p(X)}.$$

We let $\|P\|_p = \|P\|_{p,\mathbb{C}}\big(= \|P(S)\|_{\ell^p\to\ell^p}\big)$. If $1 \leqslant p < \infty$, it is easy to see that, for any complex polynomial P, we have

$$\|P\|_{p,X} = \|P(S) \otimes Id_X\|_{\ell^p_{\mathbb{Z}}(X)\to\ell^p_{\mathbb{Z}}(X)} = \|P(S_-) \otimes Id_X\|_{\ell^p(X)\to\ell^p(X)}. \tag{2.3}$$

Moreover, for all $1 \leqslant p < \infty$, by (2.2), we have

$$\|P\|_{p,S^p} \quad = \quad \|P(S)\|_{cb,\ell^p_{\mathbb{Z}}\to\ell^p_{\mathbb{Z}}}. \tag{2.4}$$

Note that, if $1 \leqslant p \leqslant \infty$, we have $\|P\|_{p,S^p} = \|P\|_{p^*,S_{p^*}}$. Moreover, if $1 \leqslant p \leqslant q \leqslant 2$, we have $\|P\|_{q,S_q} \leqslant \|P\|_{p,S^p}$ by interpolation. We define the linear map $S^p_{\mathbb{Z}} \xrightarrow{\Theta} S^p_{\mathbb{Z}}$ as the shift "from NW to SE" which maps any matrix

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ \cdots & a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ \cdots & a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

If $1 \leqslant p < \infty$, it is not difficult to see that for any complex polynomial P we have

$$\left\|P(\Theta)\right\|_{S^p_{\mathbb{Z}} \to S^p_{\mathbb{Z}}} = \left\|P(\sigma)\right\|_{S^p \to S^p} \quad \text{and} \quad \left\|P(\Theta)\right\|_{cb, S^p_{\mathbb{Z}} \to S^p_{\mathbb{Z}}} = \left\|P(\sigma)\right\|_{cb, S^p \to S^p}. \tag{2.5}$$

Moreover, it is easy to see that, for all $A \in S^p_{\mathbb{Z}}$, we have the equality $\Theta(A) = SAS^{-1}$ where we consider A and $\Theta(A)$ as operators on $\ell^2_{\mathbb{Z}}$.

We will use the following theorem inspired by a well-known technique of Kitover.

Theorem 2.2 *Suppose $1 \leqslant p \leqslant \infty$. Let X be a Banach space and $X \xrightarrow{T} X$ an isometry (not necessarily onto). For any complex polynomial P, we have the inequality*

$$\left\|P(T)\right\|_{X \to X} \leqslant \|P\|_{p,X}.$$

Proof : It suffices to consider the case $1 < p < \infty$. Let $0 < r < 1$. Since T is an isometry we have

$$\sum_{j=0}^{+\infty} \|r^j T^j(x)\|_X^p = \sum_{j=0}^{+\infty} r^{jp} \|T^j(x)\|_X^p = \|x\|_X^p \left(\sum_{j=0}^{+\infty} (r^p)^j\right) < +\infty.$$

We let $C_r = \left(\displaystyle\sum_{j=0}^{+\infty} r^{jp}\right)^{\frac{1}{p}}$. Now we define the operator

$$\begin{array}{rccc} W_r: & X & \longrightarrow & \ell^p(X) \\ & x & \longmapsto & \frac{1}{C_r}\big(x, rT(x), r^2T^2(x), \ldots, r^jT^j(x), \ldots\big) \end{array}$$

which is an isometry. If n is a positive integer and if $x \in X$ we have

$$W_r\big((rT)^n x\big) = \frac{1}{C_r}\big(r^n T^n x, r^{n+1} T^{n+1} x, \ldots\big) = (S_- \otimes Id_X)^n \big(W_r(x)\big).$$

We deduce that for any complex polynomial P we have $W_r P(rT) = P(S_- \otimes Id_X) W_r$. Now, if $x \in X$, we have

$$\begin{aligned} \left\|P(rT)x\right\|_X &= \left\|W_r\big(P(rT)x\big)\right\|_{\ell^p(X)} \\ &= \left\|P(S_- \otimes Id_X) W_r(x)\right\|_{\ell^p(X)} \end{aligned}$$

$$\begin{aligned} &\leqslant \ \left\|P(S_-) \otimes Id_X\right\|_{\ell^p(X) \to \ell^p(X)} \|x\|_X \\ &= \ \|P\|_{p,X} \|x\|_X \qquad \text{by (2.3).} \end{aligned}$$

Consequently, letting r to 1, we obtain finally that $\left\|P(T)\right\|_{X \to X} \leqslant \|P\|_{p,X}$. ■

Corollary 2.3 *Suppose* $1 \leqslant p \leqslant \infty$. *Let* P *be a complex polynomial. We have*

$$\left\|P(\sigma) \otimes Id_{S^p}\right\|_{S^p(S^p) \to S^p(S^p)} = \|P\|_{p,S^p}.$$

Proof : With the diagonal embedding of ℓ^p in S^p, we see that for any complex polynomial P we have

$$\|P\|_{p,S^p} \leqslant \left\|P(\sigma) \otimes Id_{S^p}\right\|_{S^p(S^p) \to S^p(S^p)}.$$

Now the map $S^p(S^p) \xrightarrow{\sigma \otimes Id_{S^p}} S^p(S^p)$ is an isometry. Hence, by the above theorem, we deduce that for every complex polynomial P we have

$$\left\|P(\sigma) \otimes Id_{S^p}\right\|_{S^p(S^p) \to S^p(S^p)} = \left\|P(\sigma \otimes Id_{S^p})\right\|_{S^p(S^p) \to S^p(S^p)} \leqslant \|P\|_{p,S^p(S^p)} = \|P\|_{p,S^p}.$$

■

Let M be a von Neumann algebra. Let us recall that M has QWEP means that M is the quotient of a C^*-algebra having the weak expectation property (WEP) of C. Lance (see [84] for more information on these notions). It is unknown whether every von Neumann algebra has this property. We will need the following theorem which is a particular case of a result of [50].

Theorem 2.4 *Let* M *be a von Neumann algebra with* QWEP *equipped with a faithful semifinite normal trace. Suppose* $1 < p < \infty$. *Let* Ω *be a measure space. Suppose that* $L^p(\Omega) \xrightarrow{T} L^p(\Omega)$ *is a completely bounded map. Then* $T \otimes Id_{L^p(M)}$ *extends to a bounded operator and we have*

$$\left\|T \otimes Id_{L^p(M)}\right\|_{L^p(\Omega, L^p(M)) \to L^p(\Omega, L^p(M))} \leqslant \|T\|_{cb, L^p(\Omega) \to L^p(\Omega)}.$$

In the case where M is a hyperfinite von Neumann algebra, the statement of this theorem is easy to prove (use [99, (3.1)] and [99, (3.6)]). With this theorem, we deduce the following proposition.

Proposition 2.5 *Suppose* $1 < p < \infty$. *Let* M *be a von Neumann algebra with* QWEP *equipped with a faithful semifinite normal trace. For all complex polynomial* P *we have*

$$\|P\|_{p,L^p(M)} \leqslant \|P\|_{p,S^p}.$$

With this proposition, we can prove the following corollary.

Corollary 2.6 *Let M be a von Neumann algebra equipped with a faithful semifinite normal trace and $1 < p < \infty$. Let $L^p(M) \xrightarrow{T} L^p(M)$ be a contraction. Suppose that there exists a von Neumann algebra M with QWEP equipped with a faithful semifinite normal trace, an isometric embedding $L^p(M) \xrightarrow{J} L^p(N)$, an isometry $L^p(N) \xrightarrow{U} L^p(N)$ and a contractive projection $L^p(N) \xrightarrow{Q} L^p(M)$ such that,*

$$T^k = QU^kJ$$

for any integer $k \geqslant 0$. Then the contraction T has the noncommutative Matsaev's property.

Proof : For any complex polynomial P, we have

$$\begin{aligned} \|P(T)\|_{L^p(M)\to L^p(M)} &= \|QP(U)J\|_{L^p(M)\to L^p(M)} \\ &\leqslant \|P(U)\|_{L^p(N)\to L^p(N)}. \end{aligned}$$

By using Theorem 2.2, we obtain the inequality

$$\|P(T)\|_{L^p(M)\to L^p(M)} \leqslant \|P\|_{p,L^p(N)}.$$

Now, the von Neumann algebra N is QWEP. Then, by Proposition 2.5, we obtain finally that

$$\|P(T)\|_{L^p(M)\to L^p(M)} \leqslant \|P\|_{p,S^p}.$$

■

Theorem 2.4, Proposition 2.5 and Corollary 2.6 hold true more generally for noncommutative L^p-spaces of a von Neumann algebra equipped with a distinguished normal faithful state $M \xrightarrow{\varphi} \mathbb{C}$, constructed by Haagerup. See [103] and the references therein for more informations on these spaces.

We refer to [3], [51] and [89] for information on dilations on L^p-spaces (commutative and noncommutative).

3 Comparison between the commutative and noncommutative cases

Suppose $1 < p < \infty$. Let G be a locally compact abelian group with dual group $\widehat{G}$. An operator $L^p(G) \xrightarrow{T} L^p(G)$ is a Fourier multiplier if there exists a function $\psi \in L^\infty(\widehat{G})$ such that for any $f \in L^p(G) \cap L^2(G)$ we have $\mathcal{F}(T(f)) = \psi\mathcal{F}(f)$ where $\mathcal{F}$ denotes the Fourier transform. In this case, we let $T = M_\psi$. G. Pisier showed that, if G is a compact group and $1 < p < \infty$, $p \neq 2$, there exists a bounded Fourier multiplier $L^p(G) \xrightarrow{T} L^p(G)$ which is not completely bounded (see [99, Proposition 8.1.3]. We will show this result is also true for the groups $\mathbb{R}$ and $\mathbb{Z}$ and we will prove Theorem 1.7.

If $b \in L^1(G)$, we define the convolution operator C_b by

$$\begin{array}{rccl} C_b: & L^p(G) & \longrightarrow & L^p(G) \\ & f & \longmapsto & b * f. \end{array}$$

This operator is a completely bounded Fourier multiplier. We observe that, if $P = \sum_{k=0}^n a_k z^k$ is a complex polynomial, the operator $\ell^p_{\mathbb{Z}} \xrightarrow{P(S)} \ell^p_{\mathbb{Z}}$ is the operator $\ell^p_{\mathbb{Z}} \xrightarrow{C_{\tilde{a}}} \ell^p_{\mathbb{Z}}$ where $\tilde{a}$ is the sequence defined by $\tilde{a}_k = a_k$ if $0 \leqslant k \leqslant n$ and $\tilde{a}_k = 0$ otherwise.

We will use the following approximation result [64, Theorem 5.6.1].

Theorem 3.1 *Suppose $1 \leqslant p < \infty$. Let G be a locally compact abelian group. Let $L^p(G) \xrightarrow{T} L^p(G)$ be a bounded Fourier multiplier. Then there exists a net of continuous functions $(b_l)_{l \in L}$ with compact support such that*

$$\|C_{b_l}\|_{L^p(G) \to L^p(G)} \leqslant \|T\|_{L^p(G) \to L^p(G)} \quad \text{and} \quad C_{b_i} \xrightarrow[l]{so} T$$

(convergence for the strong operator topology).

Moreover, we need the following vectorial extension of [31, Proposition 3.3]. One can prove this theorem as [24, Theorem 3.4].

Theorem 3.2 *Suppose $1 < p < \infty$. Let ψ be a continuous function on $\mathbb{R}$ which defines a completely bounded Fourier multiplier M_ψ on $L^p(\mathbb{R})$. Then the restriction $\psi|\mathbb{Z}$ of the function ψ to $\mathbb{Z}$ defines a completely bounded Fourier multiplier $M_{\psi|\mathbb{Z}}$ on $L^p(\mathbb{T})$.*

We will use the next result of Jodeit [49, Theorem 3.5]. We introduce the function $\Lambda\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\Lambda(x) = \begin{cases} 1 - |x| & \text{if} \quad x \in [-1, 1] \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

Theorem 3.3 *Suppose $1 < p < \infty$. Let φ be a complex function defined on $\mathbb{Z}$ such that M_φ is a bounded Fourier multiplier on $L^p(\mathbb{T})$. Then the complex function $\mathbb{R} \xrightarrow{\psi} \mathbb{C}$ defined on $\mathbb{R}$ by*

$$\psi(x) = \sum_{k \in \mathbb{Z}} \varphi(k) \Lambda(x - k), \qquad x \in \mathbb{R}, \tag{3.1}$$

defines a bounded Fourier multiplier $L^p(\mathbb{R}) \xrightarrow{M_\psi} L^p(\mathbb{R})$.

Now, we are ready to prove the following theorem.

Theorem 3.4 *Suppose $1 < p < \infty$, $p \neq 2$. Then there exists a bounded Fourier multiplier $L^p(\mathbb{R}) \xrightarrow{M_\psi} L^p(\mathbb{R})$ which is not completely bounded.*

Proof : By [99, Proposition 8.1.3], there exists a bounded Fourier multiplier $L^p(\mathbb{T}) \xrightarrow{M_\varphi} L^p(\mathbb{T})$ which is not completely bounded. Now, we define the function ψ on $\mathbb{R}$ by (3.1). By Theorem 3.3, the function $\mathbb{R} \xrightarrow{\psi} \mathbb{C}$ defines a bounded Fourier multiplier $L^p(\mathbb{R}) \xrightarrow{M_\psi} L^p(\mathbb{R})$. Now, suppose that M_ψ is completely bounded. Since the function $\mathbb{R} \xrightarrow{\psi} \mathbb{C}$ is continuous, by Theorem 3.2, we deduce that the restriction $\psi|\mathbb{Z}$ defines a completely bounded Fourier multiplier $M_{\psi|\mathbb{Z}}$ on $L^p(\mathbb{T})$. Moreover, we observe that, for all $k \in \mathbb{Z}$, we have

$$\psi(k) = \varphi(k).$$

Then we deduce that the Fourier multiplier $L^p(\mathbb{T}) \xrightarrow{M_\varphi} L^p(\mathbb{T})$ is completely bounded. We obtain a contradiction. Consequently, the bounded Fourier multiplier $L^p(\mathbb{R}) \xrightarrow{M_\psi} L^p(\mathbb{R})$ is not completely bounded. ■

The proof of the next theorem is inspired by [24, page 25].

Theorem 3.5 *Suppose $1 < p < \infty$, $p \neq 2$. Then*

1. *There exists a bounded Fourier multiplier $\ell^p_{\mathbb{Z}} \xrightarrow{T} \ell^p_{\mathbb{Z}}$ which is not completely bounded.*
2. *There exists a complex polynomial P such that $\|P\|_p < \|P\|_{p,S^p}$.*

Proof : By Theorem 3.4, there exists a bounded Fourier multiplier $L^p(\mathbb{R}) \xrightarrow{M_\psi} L^p(\mathbb{R})$ which is not completely bounded. We can suppose that M_ψ satisfies $\|M_\psi\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} = 1$. By Theorem 3.1, there exists a net of continuous functions $(b_l)_{l\in L}$ with compact support such that

$$\|C_{b_l}\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leqslant 1 \quad \text{and} \quad C_{b_l} \xrightarrow[l]{so} M_\psi.$$

Let $c > 1$. There exists an element $y = \sum_{k=1}^n f_k \otimes x_k \in L^p(\mathbb{R}) \otimes S^p$ with $\|y\|_{L^p(\mathbb{R},S^p)} \leqslant 1$ such that $\|(M_\psi \otimes Id_{S^p})(y)\|_{L^p(\mathbb{R},S^p)} \geqslant 3c$. Then, it is not difficult to see that there exists $l \in L$ such that $\|(C_{b_l} \otimes Id_{S^p})(y)\|_{L^p(\mathbb{R},S^p)} \geqslant 2c$. We deduce that there exists a continuous function $b \colon \mathbb{R} \to \mathbb{C}$ with compact support such that $\|C_b\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leqslant 1$ and $\|C_b\|_{cb,L^p(\mathbb{R})\to L^p(\mathbb{R})} \geqslant 2c$. Thus there exists a continuous function $\mathbb{R} \xrightarrow{b} \mathbb{C}$ with compact support such that

$$\|C_b\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leqslant 1 \qquad \text{and} \qquad \|C_b\|_{cb,L^p(\mathbb{R})\to L^p(\mathbb{R})} \geqslant 2c.$$

Now, we define the sequence $(a_n)_{n\geqslant 1}$ of complex sequences indexed by $\mathbb{Z}$ by, if $n \geqslant 1$ and $k \in \mathbb{Z}$

$$a_{n,k} = \int_0^1 \int_0^1 \frac{1}{n} b\bigg(\frac{t-s+k}{n}\bigg) ds dt.$$

Note that each sequence a_n has only a finite number of non null term. Let $n \geqslant 1$. We introduce the conditional expectation $L^p(\mathbb{R}) \xrightarrow{\mathbb{E}_n} L^p(\mathbb{R})$ with respect to the σ-algebra generated by the $\left[\frac{k}{n}, \frac{k+1}{n}\right[$,

$k \in \mathbb{Z}$. For every integer $n \geqslant 1$ and all $f \in L^p(\mathbb{R})$, we have

$$\mathbb{E}_n f = n \sum_{k \in \mathbb{Z}} \bigg(\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[}$$

(see [1, page 227]). Now, we define the linear map $\ell^p_{\mathbb{Z}} \xrightarrow{J_n} \mathbb{E}_n\big(L^p(\mathbb{R})\big)$ by, if $u \in \ell^p_{\mathbb{Z}}$

$$J_n(u) = n^{\frac{1}{p}} \sum_{k \in \mathbb{Z}} u_k 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[}.$$

It is easy to check that the map J_n is an isometry of $\ell^p_{\mathbb{Z}}$ onto the range $\mathbb{E}_n\big(L^p(\mathbb{R})\big)$ of $\mathbb{E}_n$. For any $u \in \ell^p_{\mathbb{Z}}$, we have

$$\begin{aligned}
\mathbb{E}_n C_b J_n(u) &= n \sum_{k \in \mathbb{Z}} \bigg(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \big(C_b J_n(u)\big)(t)dt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} \\
&= n \sum_{k \in \mathbb{Z}} \bigg(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{-\infty}^{+\infty} b(t-s)\big(J_n(u)\big)(s)dsdt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} \\
&= n \sum_{k \in \mathbb{Z}} \bigg(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{-\infty}^{+\infty} b(t-s) n^{\frac{1}{p}} \bigg(\sum_{j \in \mathbb{Z}} u_j 1_{\left[\frac{j}{n}, \frac{j+1}{n}\right[}(s) \bigg) dsdt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} && (3.2) \\
&= n^{1+\frac{1}{p}} \sum_{k \in \mathbb{Z}} \bigg(\sum_{j \in \mathbb{Z}} u_j \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} b(t-s)dsdt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} && (3.3) \\
&= n^{1+\frac{1}{p}} \sum_{k \in \mathbb{Z}} \bigg(\sum_{j \in \mathbb{Z}} u_j \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} b\Big(t - s + \frac{k-j}{n}\Big) dsdt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} \\
&= n^{\frac{1}{p}} \sum_{k \in \mathbb{Z}} \bigg(\sum_{j \in \mathbb{Z}} u_j \int_0^1 \int_0^1 b\Big(\frac{t-s+k-j}{n}\Big) dsdt \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} \\
&= n^{\frac{1}{p}} \sum_{k \in \mathbb{Z}} \bigg(\sum_{j \in \mathbb{Z}} u_j a_{n,k-j} \bigg) 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right[} \\
&= J_n C_{a_n}(u)
\end{aligned}$$

(where the equality (3.3) follows from the fact that the summation over $j \in \mathbb{Z}$ of (3.2) is finite). Thus we have the following commutative diagram

$$\begin{tikzcd}
L^p(\mathbb{R}) \arrow[rr, "C_b"] & & L^p(\mathbb{R}) \arrow[d, "\mathbb{E}_n"] \\
\mathbb{E}_n\big(L^p(\mathbb{R})\big) \arrow[u, hook] & & \mathbb{E}_n\big(L^p(\mathbb{R})\big) \\
\ell^p_{\mathbb{Z}} \arrow[u, "J_n", "\approx"'] \arrow[rr, "C_{a_n}"'] & & \ell^p_{\mathbb{Z}}. \arrow[u, "\approx", "J_n"']
\end{tikzcd}$$

Then, for any integer $n \geqslant 1$, since $\|\mathbb{E}_n\|_{L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})} \leqslant 1$, we have the following estimate

$$\|C_{a_n}\|_{\ell^p_{\mathbb{Z}} \longrightarrow \ell^p_{\mathbb{Z}}} \leqslant \|C_b\|_{L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})} \leqslant 1.$$

Moreover, we have $\mathbb{E}_n \otimes Id_{S^p} \xrightarrow[n \to +\infty]{so} Id_{L^p(\mathbb{R}, S^p)}$. It is easy to see that

$$(\mathbb{E}_n C_b \mathbb{E}_n) \otimes Id_{S^p} \xrightarrow[n \to +\infty]{so} C_b \otimes Id_{S^p}.$$

By the strong semicontinuity of the norm, we obtain that

$$\|C_b\|_{cb, L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})} \leqslant \liminf_{n \to \infty} \big\|\mathbb{E}_n C_b \mathbb{E}_n\big\|_{cb, L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})}.$$

Then, there exists an integer $n \geqslant 1$ such that

$$\big\|C_{a_n}\big\|_{\ell^p_{\mathbb{Z}} \longrightarrow \ell^p_{\mathbb{Z}}} \leqslant 1 \quad \text{ and } \quad \big\|C_{a_n}\big\|_{cb, \ell^p_{\mathbb{Z}} \longrightarrow \ell^p_{\mathbb{Z}}} \geqslant c.$$

Thus, we prove the second assertion by shifting the obtained multiplier. Finally, we show the first assertion by the closed graph theorem, (2.3) for $X = \mathbb{C}$ and (2.4). ∎

In the light of Corollary 2.3 and Theorem 3.5, it is natural to compare $\big\|P(\sigma)\big\|_{cb, S^p \longrightarrow S^p}$ and $\big\|P(\sigma)\big\|_{S^p \longrightarrow S^p}$. We finish the section by proving that these quantities are identical. It is a result due to É. Ricard. In order to prove it, we need the following notion of Schur multiplier. We equip $\mathbb{T}$ with its normalized Haar measure. We denote by $S^p(L^2(\mathbb{T}))$ the Schatten-von Neumann class associated with $B(L^2(\mathbb{T}))$. If $f \in L_2(\mathbb{T} \times \mathbb{T})$, we denote the associated Hilbert-Schmidt operator by

$$\begin{array}{rccc} K_f: & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & u & \longmapsto & \int_{\mathbb{T}} u(z) f(z, \cdot) dz. \end{array}$$

A Schur multiplier on $S^p(L^2(\mathbb{T}))$ is a linear map $S^p(L^2(\mathbb{T})) \xrightarrow{T} S^p(L^2(\mathbb{T}))$ such that there exists a measurable function $\mathbb{T} \times \mathbb{T} \xrightarrow{\varphi} \mathbb{C}$ which satisfies, for any finite rank operator of the form $L^2(\mathbb{T}) \xrightarrow{K_f} L^2(\mathbb{T})$, the equality $T(K_f) = K_{\varphi f}$. We denote T by M_φ and we say that the function φ is the symbol of the Schur multiplier $S^p(L^2(\mathbb{T})) \xrightarrow{M_\varphi} S^p(L^2(\mathbb{T}))$ (see [10] and [62] for more details).

We denote by $L^2(\mathbb{T}) \xrightarrow{\mathcal{F}} \ell^2_{\mathbb{Z}}$ the Fourier transform. We define the isometry Ψ by

$$\begin{array}{rccc} \Psi: & S^p(L^2(\mathbb{T})) & \longrightarrow & S^p_{\mathbb{Z}} \\ & T & \longmapsto & \mathcal{F} T \mathcal{F}^{-1}. \end{array}$$

Now, we can show the following proposition.

Proposition 3.6 *Suppose* $1 \leqslant p \leqslant \infty$. *For any complex polynomial* P, *we have*

$$\|P(\sigma)\|_{S^p \to S^p} = \|P(\sigma)\|_{cb, S^p \to S^p} \Big(= \|P(\sigma) \otimes Id_{S^p}\|_{S^p(S^p) \to S^p(S^p)} \Big).$$

Proof : It suffices to consider the case $1 < p < \infty$. For any $n \in \mathbb{Z}$ and any finite rank operator of the form K_f, we have

$$\begin{aligned}
\Big(\Theta\Psi(K_f)\Big)(e_n) &= S\mathcal{F}K_f\mathcal{F}^{-1}S^{-1}(e_n) \\
&= S\mathcal{F}K_f(z^{n-1}) \\
&= S\mathcal{F}\bigg(\int_{\mathbb{T}} z^{n-1} f(z, \cdot) dz\bigg) \\
&= S\bigg(\sum_{k \in \mathbb{Z}} \bigg(\int_{\mathbb{T}^2} z^n \overline{z'}^k f(z, z') dz dz'\bigg) e_k\bigg) \\
&= \sum_{k \in \mathbb{Z}} \bigg(\int_{\mathbb{T}^2} z^{n-1} \overline{z'}^k f(z, z') dz dz'\bigg) e_{k+1}.
\end{aligned}$$

Now we define the function $\mathbb{T} \times \mathbb{T} \xrightarrow{\varphi} \mathbb{C}$ by $\varphi(z, z') = z^{-1}z'$ where $z, z' \in \mathbb{T}$. Then, for any $n \in \mathbb{Z}$ and any finite rank operator of the form K_f, we have

$$\begin{aligned}
\Big(\Psi M_\varphi(K_f)\Big)(e_n) &= \mathcal{F}K_{\varphi f}\mathcal{F}^{-1}(e_n) \\
&= \mathcal{F}\bigg(\int_{\mathbb{T}} z^n \varphi(z, \cdot) f(z, \cdot) dz\bigg) \\
&= \sum_{k \in \mathbb{Z}} \bigg(\int_{\mathbb{T}^2} z^n \overline{z'}^k \varphi(z, z') f(z, z') dz dz'\bigg) e_k \\
&= \sum_{k \in \mathbb{Z}} \bigg(\int_{\mathbb{T}^2} z^{n-1} \overline{z'}^{k-1} f(z, z') dz dz'\bigg) e_k \\
&= \sum_{k \in \mathbb{Z}} \bigg(\int_{\mathbb{T}^2} z^{n-1} \overline{z'}^k f(z, z') dz dz'\bigg) e_{k+1}.
\end{aligned}$$

Then, for any complex polynomial P, we have the following commutative diagram

$$\begin{tikzcd}
S^p_{\mathbb{Z}} \arrow[r, "P(\Theta)"] & S^p_{\mathbb{Z}} \\
S^p(L^2(\mathbb{T})) \arrow[u, "\Psi"] \arrow[r, "P(M_\varphi)"'] & S^p(L^2(\mathbb{T})). \arrow[u, "\Psi"']
\end{tikzcd}$$

Furthermore, for any complex polynomial P, we have $P(M_\varphi) = M_{P(\varphi)}$. Moreover, the Schur multiplier

$S^p(L^2(\mathbb{T})) \xrightarrow{M_{P(\varphi)}} S^p(L^2(\mathbb{T}))$ has a continuous symbol whose the support has no isolated point. By [62, Theorem 1.19], we deduce that the norm and the completely bounded norm of $P(M_\varphi)$ coincide. Since Ψ is a complete isometry, we obtain the result by (2.5). ■

4 Positive results

Let M and N be von Neumann algebras equipped with faithful semifinite normal traces τ_M and τ_N. Let $M \xrightarrow{T} N$ a positive linear map. We say that T is trace preserving if for all $x \in L^1(M) \cap M_+$ we have $\tau_N(T(x)) = \tau_M(x)$. We will use the following straightforward extension of [53, Lemma 1.1].

Lemma 4.1 *Let M and N be von Neumann algebras equipped with faithful semifinite normal traces. Let $M \xrightarrow{T} N$ be a trace preserving unital normal positive map. Suppose $1 \leqslant p < \infty$. Then T induces a contraction $L^p(M) \xrightarrow{T} L^p(N)$. Moreover, if $M \xrightarrow{T} N$ is an one-to-one normal unital $*$-homomorphism, T induces an isometry $L^p(M) \xrightarrow{T} L^p(N)$.*

Let M be a von Neumann algebra equipped with faithful semifinite normal trace τ and N a von Neumann subalgebra such that the restriction of τ is still semifinite. Then, it is well-known that the extension $L^p(M) \xrightarrow{\mathbb{E}} L^p(N)$ of the canonical faithful normal trace preserving conditional $M \xrightarrow{\mathbb{E}} N$ is a contractive projection.

Consider the situation where $M \xrightarrow{T} M$ is a linear map such there exists a von Neumann algebra N equipped with a faithful semifinite normal trace, a unital trace preserving $*$-automorphism $N \xrightarrow{U} N$, a unital normal trace preserving one-to-one $*$-homomorphism $M \xrightarrow{J} N$ such that,

$$T^k = \mathbb{E}U^k J \tag{4.1}$$

for any integer $k \geqslant 0$, where $N \xrightarrow{\mathbb{E}} M$ is the canonical faithful normal trace preserving conditional expectation associated with J. Then, for all $1 \leqslant p < \infty$, the maps $N \xrightarrow{U} N$ and $M \xrightarrow{J} N$ extend to isometries $L_p(N) \xrightarrow{U} L_p(N)$ and $L^p(M) \xrightarrow{J} L^p(N)$ and the map $N \xrightarrow{\mathbb{E}} M$ extends to a contractive projection $L^p(N) \xrightarrow{\mathbb{E}} L^p(M)$ such that (4.1) is also true for the induced map $L^p(M) \xrightarrow{T} L^p(M)$.

In order to prove Theorems 1.5 and 1.6, we need to use fermion algebras. Since we will study maps between q-deformed algebras, we recall directly several facts about these more general algebras in the context of [18]. We denote by S_n the symmetric group. If σ is a permutation of S_n we denote by $|\sigma|$ the number $\mathrm{card}\{(i,j) \mid 1 \leqslant i,j \leqslant n, \sigma(i) > \sigma(j)\}$ of inversions of σ. Let H be a real Hilbert space with complexification $H_\mathbb{C}$. If $-1 \leqslant q < 1$ the q-Fock space over H is

$$\mathcal{F}_q(H) = \mathbb{C}\Omega \oplus \bigoplus_{n\geqslant 1} H_\mathbb{C}^{\otimes n}$$

where Ω is a unit vector, called the vacuum and where the scalar product on $H_{\mathbb{C}}^{\otimes n}$ is given by

$$\langle h_1 \otimes \cdots \otimes h_n, k_1 \otimes \cdots \otimes k_n \rangle_q = \sum_{\sigma \in S_n} q^{|\sigma|} \langle h_1, k_{\sigma(1)} \rangle_{H_{\mathbb{C}}} \cdots \langle h_n, k_{\sigma(n)} \rangle_{H_{\mathbb{C}}}.$$

If $q = -1$, we must first divide out by the null space, and we obtain the usual antisymmetric Fock space. The creation operator $l(e)$ for $e \in H$ is given by

$$\begin{array}{rccc} l(e): & \mathcal{F}_q(H) & \longrightarrow & \mathcal{F}_q(H) \\ & h_1 \otimes \cdots \otimes h_n & \longmapsto & e \otimes h_1 \otimes \cdots \otimes h_n. \end{array}$$

They satisfy the q-relation

$$l(f)^* l(e) - q l(e) l(f)^* = \langle f, e \rangle_H Id_{\mathcal{F}_q(H)}.$$

We denote by $\mathcal{F}_q(H) \xrightarrow{\omega(e)} \mathcal{F}_q(H)$ the selfadjoint operator $l(e) + l(e)^*$. The q-von Neumann algebra $\Gamma_q(H)$ is the von Neumann algebra generated by the operators $\omega(e)$ where $e \in H$. It is a finite von Neumann algebra with the trace τ defined by $\tau(x) = \langle \Omega, x.\Omega \rangle_{\mathcal{F}_q(H)}$ where $x \in \Gamma_q(H)$.

Let H and K be real Hilbert spaces and $H \xrightarrow{T} K$ be a contraction with complexification $H_{\mathbb{C}} \xrightarrow{T_{\mathbb{C}}} K_{\mathbb{C}}$. We define the following linear map

$$\begin{array}{rccc} \mathcal{F}_q(T): & \mathcal{F}_q(H) & \longrightarrow & \mathcal{F}_q(K) \\ & h_1 \otimes \cdots \otimes h_n & \longmapsto & T_{\mathbb{C}} h_1 \otimes \cdots \otimes T_{\mathbb{C}} h_n. \end{array}$$

Then there exists a unique map $\Gamma_q(H) \xrightarrow{\Gamma_q(T)} \Gamma_q(H)$ such that for every $x \in \Gamma_q(H)$ we have

$$(\Gamma_q(T)(x))\Omega = \mathcal{F}_q(T)(x\Omega).$$

This map is normal, unital, completely positive and trace preserving. If $H \xrightarrow{T} K$ is an isometry, $\Gamma_q(T)$ is an injective $*$-homomorphism. If $1 \leqslant p < \infty$, it extends to a contraction $L^p(\Gamma_q(H)) \xrightarrow{\Gamma_q(T)} L^p(\Gamma_q(K))$.

We are mainly concerned with the fermion algebra $\Gamma_{-1}(H)$. In this case, recall that if $e \in H$ has norm 1, then the operator $\omega(e)$ satisfies $\omega(e)^2 = Id_{\mathcal{F}_{-1}(H)}$. Moreover, we need the following Wick formula, (see [17, page 2] and [36, Corollary 2.1]). In order to state this, we denote, if $k \geqslant 1$ is an integer, by $\mathcal{P}_2(2k)$ the set of 2-partitions of the set $\{1, 2, \ldots, 2k\}$. If $\mathcal{V} \in \mathcal{P}_2(2k)$ we let $c(\mathcal{V})$ the number of crossings of $\mathcal{V}$, which is given, by the number of pairs of blocks of $\mathcal{V}$ which cross (see [36, page 8630] for a precise definition). Then, if $f_1, \ldots, f_{2k} \in H$ we have

$$\tau(\omega(f_1)\omega(f_2)\cdots\omega(f_{2k})) = \sum_{\mathcal{V} \in \mathcal{P}_2(2k)} (-1)^{c(\mathcal{V})} \prod_{(i,j) \in \mathcal{V}} \langle f_i, f_j \rangle_H. \tag{4.2}$$

In particular, for all $e, f \in H$, we have

$$\tau\big(\omega(e)\omega(f)\big) \quad = \quad \langle e, f\rangle_H. \tag{4.3}$$

Let $A = [a_{ij}]_{i,j\in I}$ be a matrix of $\mathbb{M}_I$. By definition, the Schur multiplier on $B(\ell^2_I)$ associated with this matrix is the unbounded linear operator M_A whose domain is the space of all $B = [b_{ij}]_{i,j\in I}$ of $B(\ell^2_I)$ such that $[a_{ij}b_{ij}]_{i,j\in I}$ belongs to $B(\ell^2_I)$, and whose action on $B = [b_{ij}]_{i,j\in I}$ is given by $M_A(B) = [a_{ij}b_{ij}]_{i,j\in I}$. For all $i, j \in I$, the matrix e_{ij} belongs to $D(M_A)$, hence M_A is densely defined for the weak* topology. Suppose $1 \leqslant p < \infty$. If for any $B \in S^p_I$, we have $B \in D(M_A)$ and the matrix $M_A(B)$ represents an element of S^p_I, by the closed graph theorem, the matrix A of $\mathbb{M}_I$ defines a bounded Schur multiplier $S^p_I \xrightarrow{M_A} S^p_I$. We have a similar statement for bounded Schur multipliers on $B(\ell^2_I)$.

Recall that a matrix A of $\mathbb{M}_I$ defines a contractive Schur multiplier $B(\ell^2_I) \xrightarrow{M_A} B(\ell^2_I)$ if and only if there exists an index set K and norm 1 vectors $h_i \in \ell^2_K$ and $k_j \in \ell^2_K$ such that for all $i, j \in I$ we have $a_{i,j} = \langle h_i, k_j\rangle_{\ell^2_K}$ (see [86]). If all entries of A are real numbers, we can take the real vector space $\ell^2_K(\mathbb{R})$ instead of the complex vector space ℓ^2_K. Finally, recall that every contractive Schur multiplier $B(\ell^2_I) \xrightarrow{M_A} B(\ell^2_I)$ is completely contractive (see [86]).

We say that a matrix A of $\mathbb{M}_I$ induces a completely positive Schur multiplier $B(\ell^2_I) \xrightarrow{M_A} B(\ell^2_I)$ if and only if for any finite set $F \subset I$ the matrix $[a_{i,j}]_{i,j\in F}$ is positive (see [86]). An other well-known characterization is that there exists vectors $h_i \in \ell^2_K(\mathbb{C})$ of norm 1 such that for all $i, j \in I$ we have $a_{i,j} = \langle h_i, h_j\rangle_{\ell^2_K}$. If A is a real matrix, we can use the real vector space $\ell^2_K(\mathbb{R})$ instead of the complex vector space ℓ^2_K.

Let M be a von Neumann algebra equipped with a semifinite normal faithful trace τ. Suppose that $M \xrightarrow{T} M$ is a normal contraction. We say that T is selfadjoint if for all $x, y \in M \cap L^1(M)$ we have

$$\tau\big(T(x)y^*\big) = \tau\big(x(Ty)^*\big).$$

In this case, it is easy to see that the restriction $T|M \cap L^1(M)$ extends to a contraction $L^1(M) \xrightarrow{T} L^1(M)$. By complex interpolation, for any $1 \leqslant p \leqslant \infty$, we obtain a contractive map $L^p(M) \xrightarrow{T} L^p(M)$. Moreover, the operator $L^2(M) \xrightarrow{T} L^2(M)$ is selfadjoint. If $M \xrightarrow{T} M$ is a normal selfadjoint complete contraction, it is easy to see that the map $L^p(M) \xrightarrow{T} L^p(M)$ is completely contractive for all $1 \leqslant p \leqslant \infty$. It is easy to see that a contractive Schur multiplier $B(\ell^2_I) \xrightarrow{M_A} B(\ell^2_I)$ associated with a matrix A of $\mathbb{M}_I$ is selfadjoint if and only if all entries of A are real.

In order to prove the next theorem, we need the following notion of infinite tensor product of von Neumann algebras, see [114]. Given a sequence $(M_n, \tau_n)_{n\in\mathbb{Z}}$ of von Neumann algebras M_n equipped with faithful normal finite traces τ_n, then on the infinite minimal C^*-tensor product of the algebras $(M_n)_{n\in\mathbb{Z}}$ there is a well-defined infinite product state $\cdots \otimes \tau_{-1} \otimes \tau_0 \otimes \tau_1 \otimes \cdots$. The weak operator closure of the GNS-representation of the infinite C^*-tensor product of $(M_n)_{n\in\mathbb{Z}}$ with respect to the state $\cdots \otimes \tau_{-1} \otimes \tau_0 \otimes \tau_1 \otimes \cdots$ yields a von Neumann algebra, called the infinite tensor product of von

Neumann algebras M_n with respect to the traces τ_n. We will denote this algebra by $\overline{\bigotimes}_{n\in\mathbb{Z}}(M_n,\tau_n)$. The state $\cdots\otimes\tau_{-1}\otimes\tau_0\otimes\tau_1\otimes\cdots$ extends to a faithful normal finite trace on $\overline{\bigotimes}_{n\in\mathbb{Z}}(M_n,\tau_n)$ which we still denote by $\cdots\otimes\tau_{-1}\otimes\tau_0\otimes\tau_1\otimes\cdots$.

The following theorem states that we can dilate some Schur multipliers. The construction (and the one of Theorem 4.6) is inspired by the work of É. Ricard [104].

Theorem 4.2 *Let $B(\ell^2_I)\xrightarrow{M_A}B(\ell^2_I)$ be a unital completely positive Schur multiplier associated with a real-valued matrix A. Then there exists a hyperfinite von Neumann algebra M equipped with a semifinite normal faithful trace, a unital trace preserving $*$-automorphism $M\xrightarrow{U}M$, a unital trace preserving one-to-one normal $*$-homomorphism $B(\ell^2_I)\xrightarrow{J}M$ such that*

$$(M_A)^k=\mathbb{E}U^kJ$$

for any integer $k\geqslant 0$, where $M\xrightarrow{\mathbb{E}}B(\ell^2_I)$ is the canonical faithful normal trace preserving conditional expectation associated with J.

Proof : Since the map $B(\ell^2_I)\xrightarrow{M_A}B(\ell^2_I)$ is completely positive we can define a positive symmetric bilinear form $\langle\cdot,\cdot\rangle_{\ell^{2,A}}$ on the real span of the e_i, where $i\in I$, by:

$$\langle e_i,e_j\rangle_{\ell^{2,A}} \quad = \quad a_{ij}. \tag{4.4}$$

We denote by $\ell^{2,A}$ the completion of the real pre-Hilbert obtained by quotient by the corresponding kernel. For all i of I we still denote by e_i the class of e_i in $\ell^{2,A}$. Now we define the von Neumann algebra M by

$$M=B(\ell^2_I)\overline{\otimes}\bigg(\overline{\bigotimes_{n\in\mathbb{Z}}}(\Gamma_{-1}(\ell^{2,A}),\tau)\bigg).$$

Since the von Neumann algebra $\Gamma_{-1}(\ell^{2,A})$ is hyperfinite, the von Neumann algebra M is also hyperfinite. We define the element d of M by

$$d=\sum_{i\in I}e_{ii}\otimes\cdots\otimes I\otimes\omega(e_i)\otimes I\otimes\cdots$$

where $\omega(e_i)$ is in position 0. Recall that M_A is unital. Then it is not difficult to see that d is a symmetry, i.e. a selfadjoint unitary element. We equip the von Neumann algebra M with the faithful semifinite normal trace $\tau_M=\mathrm{Tr}\otimes\cdots\otimes\tau\otimes\tau\otimes\cdots$. We denote by $M\xrightarrow{\mathbb{E}}B(\ell^2_I)$ the canonical faithful normal trace preserving conditional expectation of M onto $B(\ell^2_I)$. We have

$$\mathbb{E}=Id_{B(\ell^2_I)}\otimes\cdots\otimes\tau\otimes\tau\otimes\cdots.$$

We define the canonical injective normal unital $*$-homomorphism

$$\begin{array}{cccc} J: & B(\ell^2_I) & \longrightarrow & M \\ & x & \longmapsto & x \otimes \cdots \otimes I \otimes I \otimes \cdots . \end{array}$$

Clearly, J preserves the traces. We define the right shift

$$\begin{array}{cccc} \mathscr{S}: & \overline{\bigotimes}_{n\in\mathbb{Z}}(\Gamma_{-1}(\ell^{2,A}),\tau) & \longrightarrow & \overline{\bigotimes}_{n\in\mathbb{Z}}(\Gamma_{-1}(\ell^{2,A}),\tau) \\ & \cdots \otimes x_0 \otimes x_1 \otimes \cdots & \longmapsto & \cdots \otimes x_{-1} \otimes x_0 \otimes \cdots . \end{array}$$

Now, we define the linear map

$$\begin{array}{cccc} U: & M & \longrightarrow & M \\ & y & \longmapsto & d\big((Id_{B(\ell^2_I)} \otimes \mathscr{S})(y)\big)d. \end{array}$$

The map $M \xrightarrow{U} M$ is a unital $*$-automorphism of M. Moreover, it is easy to see that $M \xrightarrow{U} M$ preserves the trace τ_M. Now, we will show that, for any positive integer k, we have, for all $x \in B(\ell^2_I)$

$$U^k \circ J(x) = \sum_{i,j\in I} x_{ij}e_{ij} \otimes \cdots \otimes I \otimes \underbrace{\omega(e_i)\omega(e_j) \otimes \cdots \otimes \omega(e_i)\omega(e_j)}_{k \text{ factors}} \otimes I \otimes \cdots \tag{4.5}$$

by induction on k, where the first $\omega(e_i)\omega(e_j)$ is in position 0. The statement clearly holds for $k=0$. Now, assume (4.5). For all $x \in B(\ell^2_I)$, we have

$$\begin{aligned} U^{k+1} \circ J(x) &= d\Big((Id_{B(\ell^2_I)} \otimes \mathscr{S})\big(U^k \circ J(x)\big)\Big)d \\ &= d\Bigg((Id_{B(\ell^2_I)} \otimes \mathscr{S})\Bigg(\sum_{i,j\in I} x_{ij}e_{ij} \otimes \cdots \otimes I \otimes \omega(e_i)\omega(e_j) \otimes \cdots \otimes \omega(e_i)\omega(e_j) \otimes I \otimes \cdots\Bigg)\Bigg)d \\ &= \Bigg(\sum_{r\in I} e_{rr} \otimes \cdots \otimes I \otimes \omega(e_r) \otimes I \otimes \cdots\Bigg)\Bigg(\sum_{i,j\in I} x_{ij}e_{ij} \otimes \cdots \otimes I \otimes I \otimes \omega(e_i)\omega(e_j) \otimes \cdots \\ &\quad \otimes \omega(e_i)\omega(e_j) \otimes I \otimes \cdots\Bigg)\Bigg(\sum_{s\in I} e_{ss} \otimes \cdots \otimes I \otimes \omega(e_s) \otimes I \otimes \cdots\Bigg) \\ &= \sum_{i,j,r,s\in I} x_{ij}e_{rr}e_{ij}e_{ss} \otimes \cdots \otimes I \otimes \omega(e_r)\omega(e_s) \otimes \omega(e_i)\omega(e_j) \otimes \cdots \otimes \omega(e_i)\omega(e_j) \otimes I \otimes \cdots \\ &= \sum_{i,j\in I} x_{ij}e_{ij} \otimes \cdots \otimes I \otimes \omega(e_i)\omega(e_j) \otimes \omega(e_i)\omega(e_j) \otimes \cdots \otimes \omega(e_i)\omega(e_j) \otimes I \otimes \cdots . \end{aligned}$$

We obtained the statement (4.5) for $k+1$. Then, we deduce that for any positive integer k and any $x \in B(\ell^2_I)$ we have

$$\mathbb{E} \circ U^k \circ J(x) =$$

$$
\begin{aligned}
&= (Id_{S_I^p} \otimes \cdots \otimes \tau \otimes \cdots)\Bigg(\sum_{i,j\in I} x_{ij} e_{ij} \otimes \cdots \otimes I \otimes \omega(e_i)\omega(e_j) \otimes \cdots \otimes \omega(e_i)\omega(e_j) \otimes I \otimes \cdots\Bigg)\\
&= \sum_{i,j\in I} \tau\big(\omega(e_i)\omega(e_j)\big)^k x_{ij} e_{ij}\\
&= \sum_{i,j\in I} \big(\langle e_i, e_j\rangle_{\ell^{2,A}}\big)^k x_{ij} e_{ij} \qquad \text{by (4.3)}\\
&= \sum_{i,j\in I} (a_{ij})^k x_{ij} e_{ij} \qquad \text{by (4.4)}\\
&= (M_A)^k(x).
\end{aligned}
$$

Thus, for any positive integer k, we have

$$(M_A)^k = \mathbb{E} \circ U^k \circ J.$$

The proof is complete. ■

Corollary 4.3 *Let $B(\ell_I^2) \xrightarrow{M_A} B(\ell_I^2)$ be a contractive Schur multiplier associated with a real-valued matrix A. Suppose $1 < p < \infty$. Then, the induced Schur multiplier $S_I^p \xrightarrow{M_A} S_I^p$ satisfies the noncommutative Matsaev's property. More precisely, for any complex polynomial P, we have*

$$\big\|P(M_A)\big\|_{cb, S_I^p \to S_I^p} \leqslant \|P\|_{p, S^p}.$$

Proof : Suppose that $B(\ell_I^2) \xrightarrow{M_A} B(\ell_I^2)$ is a contractive Schur multiplier associated with a real matrix A of $\mathbb{M}_I$. There exists a set K and norm 1 vectors $h_i \in \ell_K^2(\mathbb{R})$ and $k_i \in \ell_K^2(\mathbb{R})$ such that for all $i, j \in I$ we have $a_{i,j} = \langle h_i, k_j\rangle_{\ell_K^2(\mathbb{R})}$. Now we define the following matrices of $\mathbb{M}_I$

$$B = \Big[\langle h_i, h_j\rangle_{\ell_K^2(\mathbb{R})}\Big]_{i,j\in I}, \quad C = \Big[\langle k_i, k_j\rangle_{\ell_K^2(\mathbb{R})}\Big]_{i,j\in I} \quad \text{and} \quad D = \Big[\langle k_i, h_j\rangle_{\ell_K^2(\mathbb{R})}\Big]_{i,j\in I}.$$

For all $i \in I$ and all $n \in \{1, 2\}$, we define the norm 1 vector $l_{(n,i)}$ of $\ell_K^2(\mathbb{R})$ by

$$l_{(n,i)} = \begin{cases} h_i & \text{if } n = 1 \text{ and } i \in I\\ k_i & \text{if } n = 2 \text{ and } i \in I. \end{cases}$$

Now, by the identification $\mathbb{M}_2(\mathbb{M}_I) \simeq \mathbb{M}_{\{1,2\}\times I}$, the matrix $\begin{bmatrix} B & A\\ D & C \end{bmatrix}$ of $\mathbb{M}_2(\mathbb{M}_I)$ identifies to the matrix

$$F = \Big[\langle l_{n,i}, l_{m,j}\rangle_{\ell_K^2(\mathbb{R})}\Big]_{(n,i)\in\{1,2\}\times I, (m,j)\in\{1,2\}\times I}$$

of $\mathbb{M}_{\{1,2\}\times I}$. The Schur multiplier $B(\ell^2_{\{1,2\}\times I}) \xrightarrow{M_F} B(\ell^2_{\{1,2\}\times I})$ associated with this matrix is unital and completely positive. Moreover, since the matrix F is real, $B(\ell^2_{\{1,2\}\times I}) \xrightarrow{M_F} B(\ell^2_{\{1,2\}\times I})$ is selfadjoint. Let $1 < p < \infty$. For any complex polynomial P, we have

$$\begin{aligned} \|P(M_A)\|_{S^p_I \to S^p_I} &\leqslant \left\| \begin{bmatrix} P(M_B) & P(M_A) \\ P(M_D) & P(M_C) \end{bmatrix} \right\|_{S^p_2(S^p_I) \to S^p_2(S^p_I)} \\ &= \|P(M_F)\|_{S^p_{\{1,2\}\times I} \to S^p_{\{1,2\}\times I}}. \end{aligned}$$

Now, according to Theorem 4.2, remarks following Lemma 4.1 and Corollary 2.6, the Schur multiplier M_F satisfies the noncommutative Matsaev's property. We deduce that M_A also satisfies this property. Moreover, in applying this result to the Schur multiplier $M_{I\otimes A}(= I \otimes M_A)$, we obtain the inequality for the completely bounded norm. ■

We pass to Fourier multipliers on discrete groups. Suppose that G is a discrete group. We denote by e_G the neutral element of G. We denote by $\ell^2_G \xrightarrow{\lambda(g)} \ell^2_G$ the unitary operator of left translation by g and $\mathrm{VN}(G)$ the von Neumann algebra of G spanned by the $\lambda(g)$ where $g \in G$. It is an finite algebra with trace given by

$$\tau_G(x) = \langle \varepsilon_{e_G}, x(\varepsilon_{e_G}) \rangle_{\ell^2_G}$$

where $(\varepsilon_g)_{g\in G}$ is the canonical basis of ℓ^2_G and $x \in \mathrm{VN}(G)$. A Fourier multiplier is a normal linear map $\mathrm{VN}(G) \xrightarrow{T} \mathrm{VN}(G)$ such that there exists a function $G \xrightarrow{t} \mathbb{C}$ such that for all $g \in G$ we have $T\big((\lambda(g)\big) = t_g \lambda(g)$. In this case, we denote T by

$$\begin{array}{cccc} M_t: & \mathrm{VN}(G) & \longrightarrow & \mathrm{VN}(G) \\ & \lambda(g) & \longmapsto & t_g\lambda(g). \end{array}$$

It is easy to see that a contractive Fourier multiplier $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ is selfadjoint if and only if $G \xrightarrow{t} \mathbb{C}$ is a real function. It is well-known that a Fourier multiplier $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ is completely positive if and only if the function t is positive definite. If the discrete group G is amenable, by [30, Corollary 1.8], every contractive Fourier multiplier $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ is completely contractive. Recall the following transfer result [79, Theorem 2.6].

Theorem 4.4 *Let G be an amenable discrete group. Suppose $1 < p < \infty$. Let $G \xrightarrow{t} \mathbb{R}$ a function. Let A be the matrix of $\mathbb{M}_G$ defined by $a_{g,h} = t_{gh^{-1}}$ where $g, h \in G$. The Fourier multiplier M_t is completely bounded on $L^p(\mathrm{VN}(G))$ if and only if the Schur multiplier M_A is completely bounded on S^p_G. In this case, we have*

$$\|M_t\|_{cb, L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))} = \|M_A\|_{cb, S^p_G \to S^p_G}. \tag{4.6}$$

Now, we can prove the next result.

Corollary 4.5 *Let G be an amenable discrete group. Let $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ be a contractive Fourier* *multiplier associated with a real function* $G \xrightarrow{t} \mathbb{R}$. *Suppose* $1 < p < \infty$. *Then, the induced Fourier multiplier* $L^p(\mathrm{VN}(G)) \xrightarrow{M_t} L^p(\mathrm{VN}(G))$ *satisfies the noncommutative Matsaev's property. More precisely, for any complex polynomial* P, *we have*

$$\left\| P(M_t) \right\|_{cb, L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))} \leqslant \|P\|_{p,S^p}.$$

Proof : We define the matrix A of $\mathbb{M}_G$ by $a_{g,h} = t_{gh^{-1}}$ where $g, h \in G$. By (4.6), for any complex polynomial P and all $1 < p < \infty$, we have

$$\begin{aligned} \left\| P(M_t) \right\|_{cb, L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))} &= \left\| P(M_A) \right\|_{cb, S^p_G \to S^p_G} \\ &= \left\| P(Id_{S^p} \otimes M_A) \right\|_{S^p(S^p_G) \to S^p(S^p_G)} \\ &= \left\| P(M_{I \otimes A}) \right\|_{S^p(S^p_G) \to S^p(S^p_G)}. \end{aligned}$$

Since G is amenable, the Fourier multiplier $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ is completely contractive. Moreover, the map $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ is selfadjoint. Thus, for any $1 < p < \infty$, the map $L^p(\mathrm{VN}(G)) \xrightarrow{M_t} L^p(\mathrm{VN}(G))$ is completely contractive. Then, by (4.6), for any $1 < p < \infty$, we have

$$\begin{aligned} \|M_{I \otimes A}\|_{S^p(S^p_G) \to S^p(S^p_G)} &= \|M_A\|_{cb, S^p_G \to S^p_G} \\ &= \|M_t\|_{cb, L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))} \\ &\leqslant 1. \end{aligned}$$

By Corollary 4.3, we deduce finally that, for any complex polynomial P and all $1 < p < \infty$, we have

$$\left\| P(M_t) \right\|_{L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))} \leqslant \|P\|_{p, S^p(S^p_G)} = \|P\|_{p,S^p}.$$

■

In order to prove the next theorem we need the notion of crossproduct. We refer to [110] and [112] for more information on this concept.

Theorem 4.6 *Let* G *be a discrete group. Let* $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ *be a unital completely positive Fourier multiplier associated with a real valued function* $G \xrightarrow{t} \mathbb{R}$. *Then there exists a von Neumann algebra* M *equipped with a faithful finite normal trace, a unital trace preserving* $*$*-automorphism* $M \xrightarrow{U} M$, *a unital normal trace preserving one-to-one* $*$*-homomorphism* $\mathrm{VN}(G) \xrightarrow{J} M$ *such that,*

$$(M_t)^k = \mathbb{E} U^k J$$

for any integer $k \geqslant 0$, *where* $M \xrightarrow{\mathbb{E}} VN(G)$ *is the canonical faithful normal trace preserving conditional expectation associated with* J.

Proof : Since the map $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ is completely positive, we can define a positive symmetric bilinear form $\langle \cdot, \cdot \rangle_{\ell^{2,t}}$ on the real span of the e_g, where $g \in G$, by:

$$\langle e_g, e_h \rangle_{\ell^{2,t}} = t_{g^{-1}h}.$$

We denote by $\ell^{2,t}$ the completion of the real pre-Hilbert space obtained by quotient by the corresponding kernel. For all $g \in G$, we denote by g the class of e_g in $\ell^{2,t}$. Then, for all $g, h \in G$, we have

$$\langle g, h \rangle_{\ell^{2,t}} = t_{g^{-1}h}.$$

For all $g \in G$, it easy to see that there exists a unique isometry $\ell^{2,t} \xrightarrow{\theta_g} \ell^{2,t}$ such that for all $h \in G$ we have $\theta_g(h) = gh$.

For all $g \in G$, we define the unital trace preserving $*$-automorphism $\alpha(g) = \Gamma_{-1}(\theta_g \otimes Id_{\ell^2_{\mathbb{Z}}})$:

$$\begin{array}{rccc} \alpha(g): & \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}) & \longrightarrow & \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}) \\ & w(h \otimes v) & \longmapsto & w(gh \otimes v). \end{array}$$

The homomorphism $G \xrightarrow{\alpha} \mathrm{Aut}\Big(\Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})\Big)$ allows us to define the von Neumann algebra

$$M \quad = \quad \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}) \rtimes_\alpha G. \tag{4.7}$$

We can identify $\Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})$ as a subalgebra of M. We let J the canonical normal unital injective $*$-homomorphism $\mathrm{VN}(G) \xrightarrow{J} M$. We denote by τ the faithful finite normal trace on $\Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})$. Recall that, for all $g \in G$, the map $\alpha(g)$ is trace preserving. Thus, the trace τ is α-invariant. We equip M with the induced canonical trace τ_M. For all $x \in \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})$ and all $g \in G$, we have

$$\tau_M\Big(xJ(\lambda(g))\Big) = \delta_{g,e_G}\tau(x) \tag{4.8}$$

(see [110] pages 359 and 352). If $g, h \in G$ and $v \in \ell^2_{\mathbb{Z}}$, we can write the relations of commutation of the crossed product as

$$J(\lambda(g))\omega(h \otimes v) = \omega(gh \otimes v)J(\lambda(g)). \tag{4.9}$$

We denote by $M \xrightarrow{\mathbb{E}} \mathrm{VN}(G)$ the canonical faithful normal trace preserving conditional expectation. For all $x \in \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})$ and all $g \in G$ we have

$$\mathbb{E}\Big(xJ(\lambda(g))\Big) = \tau(x)\lambda(g).$$

We define the unital trace preserving $*$-automorphism $\mathscr{S} = \Gamma_{-1}(Id_{\ell^{2,t}} \otimes S)$:

$$\begin{array}{rccc} \mathscr{S}: & \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}) & \longrightarrow & \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}) \\ & \omega(h \otimes e_n) & \longmapsto & \omega(h \otimes e_{n+1}). \end{array}$$

Since M_t is unital, $\omega(e_G \otimes e_0)$ is a symmetry, i.e. a selfadjoint unitary element. Moreover, for all $g \in G$, we have $\alpha(g)\mathscr{S} = \mathscr{S}\alpha(g)$. Then, by [112, Proposition 4.4.4], we can define a unital $*$-automorphism

$$\begin{array}{rccc} U: & M & \longrightarrow & M \\ & xJ(\lambda(g)) & \longmapsto & \omega(e_G \otimes e_0)\mathscr{S}(x)J(\lambda(g))\omega(e_G \otimes e_0). \end{array}$$

Now, we will show that U preserves the trace. For all $g \in G$ and all $x \in \Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})$, we have

$$\begin{aligned} \tau_M\Big(U\Big(xJ(\lambda(g))\Big)\Big) &= \tau_M\Big(\mathscr{S}(x)J(\lambda(g))\Big) \\ &= \delta_{g,e_G}\tau(\mathscr{S}(x)) \qquad \text{by (4.8)} \\ &= \delta_{g,e_G}\tau(x) \\ &= \tau_M\Big(xJ(\lambda(g))\Big). \end{aligned}$$

We conclude by linearity and normality. It is not hard to see that J preserves the traces.

Now, we will prove that, for any integer $k \geqslant 1$ and any $g \in G$, we have

$$\begin{aligned} & U^k \circ J(\lambda(g)) \\ &= \omega(e_G \otimes e_0)\omega(e_G \otimes e_1)\cdots\omega(e_G \otimes e_{k-1})\omega(g \otimes e_{k-1})\omega(g \otimes e_{k-2})\cdots\omega(g \otimes e_0)J(\lambda(g)) \end{aligned} \tag{4.10}$$

by induction on k. The statement holds clearly for $k = 1$: if $g \in G$, we have

$$\begin{aligned} U \circ J(\lambda(g)) &= \omega(e_G \otimes e_0)J(\lambda(g))\omega(e_G \otimes e_0) \\ &= \omega(e_G \otimes e_0)\omega(g \otimes e_0)J(\lambda(g)) \qquad \text{by (4.9).} \end{aligned}$$

Now, assume (4.10). For all $g \in G$, we have

$$\begin{aligned} & U^{k+1} \circ J(\lambda(g)) \\ &= U\Big(\omega(e_G \otimes e_0)\omega(e_G \otimes e_1)\cdots\omega(e_G \otimes e_{k-1})\omega(g \otimes e_{k-1})\omega(g \otimes e_{k-2})\cdots\omega(g \otimes e_0)J(\lambda(g))\Big) \\ &= \omega(e_G \otimes e_0)\omega(e_G \otimes e_1)\omega(e_G \otimes e_2)\cdots\omega(e_G \otimes e_k)\omega(g \otimes e_k)\omega(g \otimes e_{k-1})\cdots \\ &\quad \omega(g \otimes e_1)J(\lambda(g))\omega(e_G \otimes e_0) \\ &= \omega(e_G \otimes e_0)\omega(e_G \otimes e_1)\omega(e_G \otimes e_2)\cdots\omega(e_G \otimes e_k)\omega(g \otimes e_k)\omega(g \otimes e_{k-1})\cdots \\ &\quad \omega(g \otimes e_1)\omega(g \otimes e_0)J(\lambda(g)). \end{aligned}$$

We obtained the statement (4.10) for $k+1$. Now let $k \geqslant 1$ and $g \in G$. We define the elements $f_1, \ldots, f_{2k}$ of the Hilbert space $\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}$ by

$$f_i = \begin{cases} e_G \otimes e_{i-1} & \text{if } 1 \leqslant i \leqslant k \\ g \otimes e_{2k-i} & \text{if } k+1 \leqslant i \leqslant 2k \end{cases}$$

If $1 \leqslant i \leqslant 2k$, we have

$$\begin{aligned} \langle f_i, f_{2k-i+1} \rangle_{\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}} &= \langle e_G \otimes e_{i-1}, g \otimes e_{i-1} \rangle_{\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}} \\ &= \langle e_G, g \rangle_{\ell^{2,t}} \langle e_{i-1}, e_{i-1} \rangle_{\ell^2_{\mathbb{Z}}} \\ &= t_g \end{aligned}$$

By a similar computation, if $1 \leqslant i < j \leqslant 2k$ with $j \neq 2k-i+1$, we obtain $\langle f_i, f_j \rangle_{\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}} = 0$. Then, for all $g \in G$, we have

$$\begin{aligned} \mathbb{E}U^k J(\lambda(g)) &= \mathbb{E}\Big(\omega(e_G \otimes e_0) \cdots \omega(e_G \otimes e_{k-1}) \omega(g \otimes e_{k-1}) \cdots \omega(g \otimes e_0) J(\lambda(g))\Big) \\ &= \tau\big(\omega(e_G \otimes e_0) \cdots \omega(e_G \otimes e_{k-1}) \omega(g \otimes e_{k-1}) \cdots \omega(g \otimes e_0)\big)\lambda(g) \\ &= \tau\big(\omega(f_1)\omega(f_2) \cdots \omega(f_{2k})\big)\lambda(g) \\ &= \Bigg(\sum_{\mathcal{V} \in \mathcal{P}_2(2k)} (-1)^{c(\mathcal{V})} \prod_{(i,j) \in \mathcal{V}} \langle f_i, f_j \rangle_{\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}}\Bigg)\lambda(g) \qquad \text{by (4.2)} \qquad (4.11) \\ &= \langle f_1, f_{2k} \rangle_{\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}} \cdots \langle f_k, f_{k+1} \rangle_{\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}}} \lambda(g) \\ &= (t_g)^k \lambda(g) \\ &= T^k(\lambda(g)). \end{aligned}$$

(The only non null term in the sum of (4.11) is the term with $\mathcal{V} = \{(1,2k), (2,2k-1), \ldots, (k,k+1)\}$, which satisfies $c(\mathcal{V}) = 0$). Thus, for any positive integer k (the case $k=0$ is trivial), we conclude that

$$T^k = \mathbb{E}U^k J.$$

■

Corollary 4.7 *Let G be a discrete group. Let $\mathrm{VN}(G) \xrightarrow{M_t} \mathrm{VN}(G)$ be a unital completely positive Fourier multiplier which is associated with a real function $G \xrightarrow{t} \mathbb{R}$. Suppose that the von Neumann algebra (4.7) has QWEP. Let $1 \leqslant p \leqslant \infty$. Then, the induced Fourier multiplier $L^p(\mathrm{VN}(G)) \xrightarrow{M_t} L^p(\mathrm{VN}(G))$ satisfies the noncommutative Matsaev's property.*

Proof : This corollary follows from Theorem 4.6, remarks following Lemma 4.1 and Corollary 2.6. ■

At the light of above corollary, it is important to know when the von Neumann algebra (4.7) has QWEP. If the group G is amenable, this algebra has QWEP by [84, Proposition 4.1] (or [25, Proposition 4.8]). Now we give an example of non-amenable group G such that this von Neumann algebra has QWEP. We denote by $\mathbb{F}_n$ a free group with n generators denoted by $g_1, \ldots, g_n$ where $1 \leqslant n \leqslant \infty$. We denote by $\mathcal{R}$ the hyperfinite factor of type II_1 and by $\mathcal{R}^{\mathcal{U}}$ an ultrapower of $\mathcal{R}$ with respect to a non-trivial ultrafilter $\mathcal{U}$. In order to prove the next theorem we need the notion of amalgamated free product of von Neumann algebras. We refer to [12] and [115] for more information on this concept. Note that, with the notations of the proof of Theorem 4.6, the von Neumann algebra $\Gamma_{-1}(\ell^{2,t} \otimes_2 \ell^2_{\mathbb{Z}})$ is $*$-isomorphic to the hyperfinite factor of type II_1.

Proposition 4.8 *Suppose* $1 \leqslant n \leqslant \infty$. *Let* $\mathbb{F}_n \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{R})$ *be a homomorphism. Then the crossed product* $\mathcal{R} \rtimes_\alpha \mathbb{F}_n$ *has QWEP.*

Proof : First we will show the result for $n = 2$. We denote by $\langle g_1 \rangle$ and $\langle g_2 \rangle$ the subgroups of $\mathbb{F}_2$ generated by g_1 and g_2 and by α_1 and α_2 the restrictions of α to these subgroups. First, we prove that the subalgebras $\mathcal{R} \rtimes_{\alpha_1} \langle g_1 \rangle$ and $\mathcal{R} \rtimes_{\alpha_2} \langle g_2 \rangle$ are free with respect to the canonical faithful normal trace preserving conditional expectation $\mathcal{R} \rtimes_\alpha \mathbb{F}_2 \xrightarrow{\mathbb{E}} \mathcal{R}$. We identify $\mathcal{R}$ as a subalgebra of $\mathcal{R} \rtimes_\alpha \mathbb{F}_2$. We may regard the elements of $\mathcal{R} \rtimes_\alpha \mathbb{F}_2$ as matrices $\big[\alpha_{r^{-1}}(\varpi(rt^{-1}))\big]_{r,t \in \mathbb{F}_2}$ with entries in $\mathcal{R}$ where $\mathbb{F}_2 \xrightarrow{\varpi} \mathcal{R}$ is a map. Recall that the conditional expectation $\mathbb{E}$ on $\mathcal{R}$ is given by

$$\mathbb{E}\Big(\big[\alpha_{r^{-1}}(\varpi(rt^{-1}))\big]_{r,t \in \mathbb{F}_2}\Big) = \varpi(e_{\mathbb{F}_2}).$$

Suppose that $i_1, \ldots, i_k \in \{1, 2\}$ are integers such that $i_1 \neq i_2, \ldots, i_{k-1} \neq i_k$. For any $1 \leqslant j \leqslant k$, let

$$A_j = \big[\alpha_{r^{-1}}(\varpi_j(rt^{-1}))\big]_{r,t \in \mathbb{F}_2}$$

be an element of $\mathcal{R} \rtimes_{\alpha_{i_j}} \langle g_{i_j} \rangle$ such that $\mathbb{E}(A_j) = 0$ where each $\mathbb{F}_2 \xrightarrow{\varpi_j} \mathcal{R}$ is a map satisfying $\varpi_j(g) = 0$ if $g \notin \langle g_{i_j} \rangle$. Then, for all $1 \leqslant j \leqslant k$, we have $\varpi_j(e_{\mathbb{F}_2}) = 0$. Now, we have

$$\begin{aligned}
\mathbb{E}(A_1 \cdots A_k) &= \mathbb{E}\Big(\big[\alpha_{r^{-1}}(\varpi_1(rt^{-1}))\big]_{r,t \in \mathbb{F}_2} \cdots \big[\alpha_{r^{-1}}(\varpi_k(rt^{-1}))\big]_{r,t \in \mathbb{F}_2}\Big) \\
&= \sum_{L^1, \ldots, l_{k-1} \in \mathbb{F}_2} \varpi_1(l_1^{-1}) \alpha_{l_1^{-1}}\big(\varpi_2(l_1 l_2^{-1})\big) \cdots \alpha_{l_{k-2}^{-1}}\big(\varpi_2(l_{k-2} l_{k-1}^{-1})\big) \alpha_{l_{k-1}^{-1}}(\varpi_k(l_{k-1})) \\
&= 0.
\end{aligned}$$

Thus the von Neumann algebra $\mathcal{R} \rtimes_\alpha \mathbb{F}_2$ decomposes as an amalgamated free product of $\mathcal{R} \rtimes_{\alpha_1} \langle g_1 \rangle$ and $\mathcal{R} \rtimes_{\alpha_2} \langle g_2 \rangle$ over $\mathcal{R}$. Moreover, the groups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are commutative, hence amenable. We have already point out that the crossed product of the hyperfinite factor $\mathcal{R}$ by an amenable group has QWEP. Then the von Neumann algebras $\mathcal{R} \rtimes_{\alpha_1} \langle g_1 \rangle$ and $\mathcal{R} \rtimes_{\alpha_2} \langle g_2 \rangle$ are QWEP. Moreover, by [11, page 283], these von Neumann algebras have a separable predual. By [57, Theorem 1.4], we deduce that these von Neumann algebras are embeddable into $\mathcal{R}^{\mathcal{U}}$. Now, the theorem stating in [20, Corollary 4.5] says that, for finite von Neumann algebras with separable preduals, being embeddable into $\mathcal{R}^{\mathcal{U}}$ is stable under amalgamated free products over a hyperfinite von Neumann algebra. Thus we deduce that $\mathcal{R}\rtimes_\alpha \mathbb{F}_2$ is embeddable into $\mathcal{R}^{\mathcal{U}}$, which is equivalent to QWEP, by [57, Theorem 1.4], since $\mathcal{R}\rtimes_\alpha \mathbb{F}_2$ has a separable predual. Induction then gives the case when $2 \leqslant n < \infty$, and the case $n = \infty$ then follows since, by [84, Proposition 4.1], QWEP is preserved by taking the weak* closure of increasing unions of von Neumann algebras. ■

We pass to maps arising in the second quantization in the context of [18].

Proposition 4.9 *Suppose $1 < p < \infty$ and $-1 \leqslant q < 1$. Let H be a real Hilbert space and $H \xrightarrow{T} H$ a contraction. Then the induced map $L^p(\Gamma_q(H)) \xrightarrow{M_t} L^p(\Gamma_q(H))$ satisfies the noncommutative Matsaev's property.*

Proof : There exists an orthogonal dilation $K \xrightarrow{U} K$ de $H \xrightarrow{T} H$. We denote by $H \xrightarrow{J} K$ the embedding of H in K and $K \xrightarrow{Q} H$ the projection of K on H. The map $\Gamma_q(K) \xrightarrow{\Gamma(J)} \Gamma_q(K)$ is a unital injective normal trace preserving $*$-homomorphism. The map $\Gamma_q(H) \xrightarrow{\Gamma(U)} \Gamma_q(K)$ is a unital trace preserving $*$-automorphism. The map $\Gamma_q(K) \xrightarrow{\Gamma(Q)} \Gamma_q(H)$ is the canonical faithful normal unital trace preserving conditional expectation of $\Gamma_q(K)$ on $\Gamma_q(H)$. Moreover, we have for any integer k

$$\Gamma_q(T)^k = \Gamma_q(P)\Gamma_q(U)^k\Gamma_q(J).$$

We conclude with Theorem 4.6, remarks following Lemma 4.1, Corollary 2.6 and by using the fact that, by [83], the von Neumann algebra $\Gamma_q(H)$ has QWEP. ■

In order to state more easily our following result we need to define the following property. Let M be a von Neumann algebra. Suppose that $M \xrightarrow{T} M$ is a linear map.

Property 4.10 *There exists a von Neumann algebra N with QWEP equipped with a normal faithful finite trace on N, a unital trace preserving $*$-automorphism $N \xrightarrow{U} N$, a unital injective normal trace preserving $*$-homomorphism $M \xrightarrow{J} N$ such that,*

$$T^k = \mathbb{E}U^k J.$$

for any integer $k \geqslant 0$, where $M \xrightarrow{\mathbb{E}} VN(G)$ is the canonical faithful normal trace preserving conditional expectation associated with J.

This property is stable under free product. Indeed, one can prove the next proposition with an argument similar to that used in the proof of [52, Lemma 10.4] and by using [20, Corollary 4.5] and [57, Theorem 1.4].

Proposition 4.11 *Let M_1 and M_2 be von Neumann algebras with separable preduals equipped with normal faithful finite traces τ_1 and τ_2. Let $M_1 \xrightarrow{T_1} M_1$ and $M_2 \xrightarrow{T_2} M_2$ be linear maps. If T_1 and T_2 satisfy Property 4.10, their free product*

$$(M_1,\tau_1)\overline{*}(M_2,\tau_2) \xrightarrow{T_1\overline{*}T_2} (M_1,\tau_1)\overline{*}(M_2,\tau_2)$$

also satisfies Property 4.10.

Thus the above proposition allows us to construct other examples of contractions satisfying the noncommutative Matsaev's property.

5 The case of semigroups

Suppose $1 \leqslant p < \infty$. We denote by $(\mathfrak{T}_t)_{t\geqslant 0}$ the translation semigroup on $L^p(\mathbb{R})$, where $\mathfrak{T}_t(f)(s) = f(s-t)$ if $f \in L^p(\mathbb{R})$ and $s,t \in \mathbb{R}$. This semigroup $(\mathfrak{T}_t)_{t\geqslant 0}$ is a C_0-semigroup of contractions.

Let $(T_t)_{t\geqslant 0}$ be a C_0-semigroup of contractions on a Banach space X. For all $b \in L^1(\mathbb{R})$ with support in $\mathbb{R}^+$, it is easy to see that the linear operator

$$\begin{array}{rccc} \int_0^{+\infty} b(t)T_t dt: & X & \longrightarrow & X \\ & x & \longmapsto & \int_0^{+\infty} b(t)T_t x dt \end{array}$$

is well-defined and bounded. Moreover, we have

$$\left\| \int_0^{+\infty} b(t)T_t dt \right\|_{X\longrightarrow X} \leqslant \|b\|_{L^1(\mathbb{R})}.$$

Now, let us state a question for semigroups which is analogue to Matsaev's Conjecture 1.1.

Question 5.1 *Suppose $1 < p < \infty$, $p \neq 2$. Let $(T_t)_{t\geqslant 0}$ be a C_0-semigroup of contractions on a L^p-space $L^p(\Omega)$ of a measure space Ω. Do we have the following estimate*

$$\left\| \int_0^{+\infty} b(t)T_t dt \right\|_{L^p(\Omega)\longrightarrow L^p(\Omega)} \leqslant \left\| \int_0^{+\infty} b(t)\mathfrak{T}_t dt \right\|_{L^p(\mathbb{R})\longrightarrow L^p(\mathbb{R})} \tag{5.1}$$

for all $b \in L^1(\mathbb{R})$ with support in $\mathbb{R}^+$?

We pass to the noncommutative case. We can state the following noncommutative analogue of Question 5.1.

Question 5.2 *Suppose $1 < p < \infty$, $p \neq 2$. Let $(T_t)_{t\geqslant 0}$ be a C_0-semigroup of contractions on a noncommutative L^p-space $L^p(M)$. Do we have the following estimate*

$$\left\| \int_0^{+\infty} b(t)T_t dt \right\|_{L^p(M)\longrightarrow L^p(M)} \leqslant \left\| \int_0^{+\infty} b(t)\mathfrak{T}_t dt \right\|_{cb,L^p(\mathbb{R})\longrightarrow L^p(\mathbb{R})} \tag{5.2}$$

for all $b \in L^1(\mathbb{R})$ with support in $\mathbb{R}^+$?

For all $b \in L^1(\mathbb{R})$ with support in $\mathbb{R}^+$, it is clear that $C_b = \int_0^{+\infty} b(t)\mathfrak{T}_t dt$. Moreover, for all $b \in L^1(\mathbb{R})$, we have

$$\|C_b\|_{L^1(\mathbb{R})\longrightarrow L^1(\mathbb{R})} = \|C_b\|_{cb,L^1(\mathbb{R})\longrightarrow L^1(\mathbb{R})} = \|b\|_{L^1(\mathbb{R})}.$$

Consequently, the inequalities (5.1) and (5.2) hold true for $p = 1$.

In [24, page 25], it is proved that the C_0-semigroups of positive contractions satisfy inequality (5.1). Using [89, Theorem 3] and the same method, we can generalize this result to C_0-semigroups of operators which admit a contractive majorant. Now, we adapt this method in order to give a link between Question 5.2 and Question 1.3.

Theorem 5.3 *Suppose $1 < p < \infty$. Let $(T_t)_{t \geqslant 0}$ be a C_0-semigroup of contractions on a noncommutative L^p-space $L^p(M)$ such that each $L^p(M) \xrightarrow{T_t} L^p(M)$ satisfies the noncommutative Matsaev's property. Then the semigroup $(T_t)_{t \geqslant 0}$ satisfies inequality (5.2).*

Proof : It is not hard to see that it suffices to prove this in the case when b has compact support. Now we define the sequence $(a_n)_{n \geqslant 1}$ of complex sequences indexed by $\mathbb{Z}$ as in the proof of Theorem 3.5. Let $n \geqslant 1$. Observe that if $\mathbb{R}^+ \xrightarrow{f} L^p(M)$ is continuous and piecewise affine with nodes at $\frac{k}{n}$ then

$$\int_0^{+\infty} b(t) f(t) dt = \sum_{k=0}^{+\infty} a_{n,k} f\left(\frac{k}{n}\right).$$

Let $x \in L^p(M)$. Let $\mathbb{R}^+ \xrightarrow{f_n} L^p(M)$ be the continuous and piecewise affine function with nodes at $\frac{k}{n}$ such that $f_n(\frac{k}{n}) = (T_{\frac{1}{n}})^k x$. Since the map $t \mapsto T_t x$ is uniformly continuous on compacts of $\mathbb{R}^+$ we have

$$\begin{aligned}
\left\| \int_0^{+\infty} b(t) T_t x dt - \sum_{k=0}^{+\infty} a_{n,k} (T_{\frac{1}{n}})^k x \right\|_{L^p(M)} &= \left\| \int_0^{+\infty} b(t) T_t x dt - \sum_{k=0}^{+\infty} a_{n,k} f_n\left(\frac{k}{n}\right) \right\|_{L^p(M)} \\
&= \left\| \int_0^{+\infty} b(t) \big(T_t x - f_n(t)\big) dt \right\|_{L^p(M)} \xrightarrow[n \to +\infty]{} 0.
\end{aligned}$$

We deduce that

$$\sum_{k=0}^{+\infty} a_{n,k} (T_{\frac{1}{n}})^k \xrightarrow[n \to +\infty]{so} \int_0^{+\infty} b(t) T_t dt.$$

By the commutative diagram of the proof of Theorem 3.5, we have for any integer $n \geqslant 1$

$$\|C_{a_n}\|_{cb, \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}} \leqslant \|C_b\|_{cb, L^p(\mathbb{R}) \to L^p(\mathbb{R})}.$$

Finally, by the strongly lower semicontinuity of the norm, we obtain that

$$\begin{aligned}
\left\| \int_0^{+\infty} b(t) T_t \, dt \right\|_{L^p(M) \to L^p(M)} &\leqslant \liminf_{n \to +\infty} \left\| \sum_{k=1}^{+\infty} a_{n,k} (T_{\frac{1}{n}})^k \right\|_{L^p(M) \to L^p(M)} \\
&\leqslant \liminf_{n \to +\infty} \|C_{a_n}\|_{cb, \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}} \\
&= \|C_b\|_{cb, L^p(\mathbb{R}) \to L^p(\mathbb{R})}.
\end{aligned}$$

■

The first consequence of this theorem is that inequality (5.2) holds true for $p = 2$. Now, we list some natural examples of semigroups which satisfy inequality (5.2) by our results, using this theorem.

Semigroups of Schur multipliers. Let $(T_t)_{t \geqslant 0}$ a w*-semigroup of selfadjoint contractive Schur multipliers on $B(\ell^2_I)$. If $1 \leqslant p < \infty$ and $t \geqslant 0$, the map $B(\ell^2_I) \xrightarrow{T_t} B(\ell^2_I)$ induces a contraction $S^p_I \xrightarrow{T_t} S^p_I$. Using [52, Remark 5.2], it is easy to see that we obtain a C_0-semigroup of contractions $S^p_I \xrightarrow{T_t} S^p_I$ which satisfies inequality (5.2).

Semigroups of Fourier multipliers on an amenable group. Let G be an amenable group. Let $(T_t)_{t \geqslant 0}$ a w*-semigroup of selfadjoint contractive Fourier multipliers on $VN(G)$. If $1 \leqslant p < \infty$ and $t \geqslant 0$, the map $VN(G) \xrightarrow{T_t} VN(G)$ induces a contraction $L^p(VN(G)) \xrightarrow{T_t} L^p(VN(G))$. We obtain a C_0-semigroup of contractions $L^p(VN(G)) \xrightarrow{T_t} L^p(VN(G))$ which satisfies inequality (5.2).

Noncommutative Poisson semigroup. Let $n \geqslant 1$ be an integer. Recall that $\mathbb{F}_n$ denotes a free group with n generators denoted by $g_1, \ldots, g_n$. A semigroup on $L^p(VN(\mathbb{F}_n))$ induced by a w^*-semigroup of selfadjoint completely positive unital Fourier multipliers on $VN(\mathbb{F}_n)$ satisfies inequality (5.2). An example is provided by the following semigroup. Any $g \in \mathbb{F}_n$ has a unique decomposition of the form

$$g = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_l}^{k_l},$$

where $l \geqslant 0$ is an integer, each i_j belongs to $\{1, \ldots, n\}$, each k_j is a non zero integer, and $i_j \neq i_{j+1}$ if $1 \leqslant j \leqslant l-1$. The case when $l = 0$ corresponds to the unit element $g = e_{\mathbb{F}_n}$. By definition, the length of g is defined as

$$|g| = |k_1| + \cdots + |k_l|.$$

This is the number of factors in the above decomposition of g. For any nonnegative real number $t \geqslant 0$, we have a normal unital completely positive selfadjoint map

$$\begin{array}{cccc} T_t : & \mathrm{VN}(\mathbb{F}_n) & \longrightarrow & \mathrm{VN}(\mathbb{F}_n) \\ & \lambda(g) & \longmapsto & e^{-t|g|}\lambda(g). \end{array}$$

These maps define a w*-semigroup $(T_t)_{t \geqslant 0}$ called the noncommutative Poisson semigroup (see [52] for more information). If $1 \leqslant p < \infty$, this semigroup defines a C_0-semigroup of contractions $L^p(\mathrm{VN}(\mathbb{F}_n)) \xrightarrow{T_t} L^p(\mathrm{VN}(\mathbb{F}_n))$ which satisfies inequality (5.2).

q-Ornstein-Uhlenbeck semigroup. Suppose $-1 \leqslant q < 1$. Let H be a real Hilbert space and let $(a_t)_{t \geqslant 0}$ be a C_0-semigroup of contractions on H. For any $t \geqslant 0$, let $T_t = \Gamma_q(a_t)$. Then $(T_t)_{\geqslant 0}$ is a w*-semigroup of normal unital completely positive maps on the von Neumann algebra $\Gamma_q(H)$. If

$1 \leqslant p < \infty$, this semigroup defines a C_0-semigroup of contractions $L^p\big(\Gamma_q(H)\big) \xrightarrow{T_t} L^p\big(\Gamma_q(H)\big)$ (see [52] for more information). This semigroup satisfies inequality (5.2).

In the case where $a_t = e^{-t} I_H$, the semigroup $(T_t)_{\geqslant 0}$ is the so-called q-Ornstein-Uhlenbeck semigroup.

Modular semigroups. The C_0-semigroups of isometries satisfy inequality (5.2). Examples are provided by modular automorphisms semigroups. Here we use noncommutative L^p-spaces of a von Neumann algebra equipped with a distinguished normal faithful state, constructed by Haagerup. We refer to [103], and the references therein, for more information on these spaces. Let M be a von Neumann algebra with QWEP equipped with a normal faithful state $M \xrightarrow{\varphi} \mathbb{C}$. Let $\big(\sigma_t^\varphi\big)_{t\in\mathbb{R}}$ be the modular group of φ. If $1 \leqslant p < \infty$, it is well known that $\big(\sigma_t^\varphi\big)_{t\geqslant 0}$ induces a C_0-semigroup of isometries $L^p(M) \xrightarrow{\sigma_t^\varphi} L^p(M)$ (see [53]). This semigroup satisfies inequality (5.2).

In the light of Theorem 4.2, it is natural to ask for dilations of unital selfadjoint completely positive semigroups of Schur multipliers. Actually, these semigroups admit a description which allows us to construct a such dilation.

Proposition 5.4 *Suppose that A is a matrix of $\mathbb{M}_I$. For all $t \geqslant 0$, let T_t be the unbounded Schur multipliers on $B\big(\ell^2_I\big)$ associated with the matrix*

$$\Big[e^{-ta_{ij}}\Big]_{i,j\in I}. \tag{5.3}$$

Then the semigroup $(T_t)_{t\geqslant 0}$ extends to a semigroup of selfadjoint unital completely positive Schur multipliers $B\big(\ell^2_I\big) \xrightarrow{T_t} B\big(\ell^2_I\big)$ if and only if there exists a Hilbert space H and a family $(\alpha_i)_{i\in I}$ of elements of H such that for all $t \geqslant 0$ the Schur multiplier $B\big(\ell^2_I\big) \xrightarrow{T_t} B\big(\ell^2_I\big)$ is associated with the matrix

$$\Big[e^{-t\|\alpha_i-\alpha_j\|_H^2}\Big]_{i,j\in I}.$$

In this case, the Hilbert space may be chosen as a real Hilbert space. Moreover, $(T_t)_{t\geqslant 0}$ is a w^*-semigroup.

Proof : Now say that each T_t is a selfadjoint unital completely positive contraction means that for all $t > 0$, the matrix (5.3) defines a real-valued positive definite kernel on $I \times I$ in the sense of [6, Chapter 3, Definition 1.1] such that for all $i \in I$ we have $a_{ii} = 0$. Now, the theorem of Schoenberg [6, Theorem 2.2] affirms that if ψ is a kernel then $e^{-t\psi}$ is a positive definite kernel for all $t > 0$ if and only if ψ is a negative definite kernel. Consequently, the last assertion is equivalent to the fact that A defines a real-valued negative definite kernel which vanishes on the diagonal of $I \times I$. Finally, the characterization of real-valued definite negative kernel of [6, Proposition 3.2] gives the equivalence with the required description.

The assertion concerning the choice of the Hilbert space is clear. Finally, using [52, Remark 5.2], it is easy to see that $(T_t)_{t\geqslant 0}$ is a w*-semigroup. ■

The next proposition is inspired by the work [53].

Proposition 5.5 *Let $(T_t)_{t\geqslant 0}$ be a w*-semigroup of selfadjoint unital completely positive Schur multipliers on $B(\ell^2_I)$. Then, there exists a hyperfinite von Neumann algebra M equipped with a semifinite normal faithful trace, a w*-semigroup $(U_t)_{t\geqslant 0}$ of unital trace preserving *-automorphisms of M, a unital trace preserving one-to-one normal $*$-homomorphism $B(\ell^2_I) \xrightarrow{J} M$ such that*

$$T_t = \mathbb{E}U_t J.$$

for any $t \geqslant 0$, where $M \xrightarrow{\mathbb{E}} B(\ell^2_I)$ is the canonical faithful normal trace preserving conditional expectation associated with J.

Proof : By Proposition 5.4, there exists a real Hilbert space H and a family $(\alpha_j)_{j\in I}$ of elements of H such that, for all $t \geqslant 0$, the Schur multiplier $B(\ell^2_I) \xrightarrow{T_t} B(\ell^2_I)$ is associated with the matrix

$$\Big[e^{-t\|\alpha_j-\alpha_k\|^2_H}\Big]_{j,k\in I}.$$

Let μ be a gaussian measure on H, i.e. a probability space (Ω,μ) together with a measurable function $\Omega \xrightarrow{w} H$ such that, for all $h \in H$, we have

$$e^{-\|h\|^2_H} = \int_\Omega e^{i\langle h,w(\omega)\rangle_H} d\mu(\omega)$$

where $i^2 = -1$. We define the von Neumann algebra $M = L^\infty(\Omega)\overline{\otimes}B(\ell^2_I)$. Note that M is a hyperfinite von Neumann algebra. We equip the von Neumann algebra M with the faithful semifinite normal trace $\tau_M = \int_\Omega \cdot\, d\mu \otimes \mathrm{Tr}$. Note that, by [107, Theorem 1.22.13], we have a $*$-isomorphism $M = L^\infty(\Omega, B(\ell^2_I))$. We define the canonical injective normal unital $*$-homomorphism

$$\begin{array}{cccc} J: & B(\ell^2_I) & \longrightarrow & L^\infty(\Omega)\overline{\otimes}B(\ell^2_I) \\ & x & \longmapsto & 1\otimes x. \end{array}$$

It is clear that the map J preserves the traces. We denote by $M \xrightarrow{\mathbb{E}} B(\ell^2_I)$ the canonical faithful normal trace preserving conditional expectation of M onto $B(\ell^2_I)$. For all $\omega \in \Omega$ and $t > 0$ let $D_t(\omega)$ be the diagonal matrix of $B(\ell^2_I)$ defined by

$$D_t(\omega) = \Big[\delta_{j,k} e^{i\sqrt{t}\langle \alpha_j, w(\omega)\rangle_{\ell^2_I}}\Big]_{j,k\in I}.$$

Note that, for all $t > 0$, the map $\Omega \xrightarrow{D_t} B(H)$ defines an unitary element of $L^\infty(\Omega, B(\ell^2_I))$. Now, for all $t \geqslant 0$ we define the linear map

$$\begin{array}{rccc} U_t: & L^\infty(\Omega, B(\ell^2_I)) & \longrightarrow & L^\infty(\Omega, B(\ell^2_I)) \\ & f & \longmapsto & D_t f D_t^*. \end{array}$$

If $t \geqslant 0$, it is easy to see that the map U_t is a trace preserving $*$-automorphism of M. For all $x \in B(\ell^2_I)$, we have

$$\begin{aligned} \mathbb{E}U_t J(x) &= \mathbb{E}U_t(1 \otimes x) \\ &= \int_\Omega D_t(\omega)(1 \otimes x) D_t(\omega)^* d\mu(\omega) \\ &= \int_\Omega \Big[e^{i\sqrt{t}\langle \alpha_j - \alpha_k, w(\omega)\rangle_H} x_{jk} \Big]_{j,k \in I} d\mu(\omega) \\ &= \Big[e^{-t\|\alpha_j - \alpha_k\|_H^2} x_{jk} \Big]_{j,k \in I} \\ &= T_t(x). \end{aligned}$$

Thus, for all $t \geqslant 0$, we have

$$T_t = \mathbb{E}U_t J.$$

The assertion concerning the regularity of the semigroup is easy and left to the reader. ■

In the same vein, it is not difficult to construct a dilation of the noncommutative Poisson semigroup. It is an unpublished result of F. Lust-Piquard. Moreover, it is easy to dilate the C_0-semigroups of contractions $L^p(\Gamma_q(H)) \xrightarrow{\Gamma_q(a_t)} L^p(\Gamma_q(H))$, with [113, Theorem 8.1].

Finally, we have the next result analogue to Corollary 2.6. One can prove this proposition with a similar argument.

Proposition 5.6 *Suppose $1 < p < \infty$. Let $(T_t)_{t \geqslant 0}$ be a C_0-semigroup of contractions on a noncommutative L^p-space $L^p(M)$. Suppose that there exists a noncommutative L^p-space $L^p(N)$ where N has QWEP, a C_0-semigroup $(U_t)_{t \geqslant 0}$ of isometric operators on $L^p(N)$, an isometric embedding $L^p(M) \xrightarrow{J} L^p(N)$ and a contractive map $L^p(N) \xrightarrow{Q} L^p(M)$ such that,*

$$T_t = QU_t J.$$

for any $t \geqslant 0$. Then, for all $b \in L^1(\mathbb{R})$ with support in $\mathbb{R}^+$, we have the estimate

$$\left\| \int_0^{+\infty} b(t) T_t dt \right\|_{L^p(M) \longrightarrow L^p(M)} \leqslant \left\| \int_0^{+\infty} b(t) \mathfrak{T}_t dt \right\|_{cb, L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})}.$$

Moreover, if $L^p(N)$ is a commutative L^p-space $L^p(\Omega)$, we have, for all $b \in L^1(\mathbb{R})$ with support in $\mathbb{R}^+$, *the estimate*

$$\left\| \int_0^{+\infty} b(t) T_t dt \right\|_{L^p(\Omega) \longrightarrow L^p(\Omega)} \leqslant \left\| \int_0^{+\infty} b(t) \mathfrak{T}_t dt \right\|_{L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})}.$$

This proposition allows us to give alternate proofs for some results of this section. By example, using [113, Theorem 8.1] of dilation of C_0-semigroups on a Hilbert space, we deduce that the both inequalities (5.1) and (5.2) are true for $p = 2$. By using [40], we see that the C_0-semigroups of operators which admit a contractive majorant satisfy inequality (5.1), for $1 < p < \infty$.

6 Final remarks

We begin by observing that the inequalities (1.1) and (1.4) are true for any complex polynomial P of degree 1 and any contraction T. Indeed, suppose that $P(z) = a + bz$, then it is easy to see that $\|P\|_2 = |a| + |b|$. Thus, for all $1 \leqslant p \leqslant \infty$, we have $\|P\|_p = \|P\|_{p,S^p} = |a| + |b|$.

Now we will determine the real polynomials of higher degree with a similar property.

Proposition 6.1 *Let* $P = \sum_{k=0}^{n} a_k z^k$ *be a real polynomial such that* $a_k \neq 0$ *for any* $0 \leqslant k \leqslant n$. *The following assertions are equivalent.*

1. *For all* $1 < p < \infty$, *we have* $\|P\|_p = \sum_{k=0}^{n} |a_k|$.
2. *For all* $1 < p < \infty$, *we have* $\|P\|_{p,S^p} = \sum_{k=0}^{n} |a_k|$.
3. *There exists* $1 < p < \infty$ *such that* $\|P\|_p = \sum_{k=0}^{n} |a_k|$.
4. *There exists* $1 < p < \infty$ *such that* $\|P\|_{p,S^p} = \sum_{k=0}^{n} |a_k|$.
5. *The coefficients* a_k *have the same sign or the signs of the* a_k *are alternating (i.e. for any integer* $0 \leqslant k \leqslant n-1$ *we have* $a_k a_{k+1} \leqslant 0$*).*

In this case, for the polynomial P *and any contraction* T, *the inequalities (1.1) and (1.4) are true.*

Proof : First we will show that $\|P\|_2 = \sum_{k=0}^{n} |a_k|$ is equivalent to the last assertion. Recall that $\|P\|_2 = \sup_{|z|=1} |P(z)|$. On the one hand, for all $0 \leqslant \theta \leqslant 2\pi$, we have

$$\begin{aligned} \left| \sum_{k=0}^{n} a_k e^{ki\theta} \right|^2 &= \left(\sum_{k=0}^{n} a_k \cos(k\theta) \right)^2 + \left(\sum_{k=0}^{n} a_k \sin(k\theta) \right)^2 \\ &= \sum_{k=0}^{n} a_k^2 \cos^2(k\theta) + 2 \sum_{0 \leqslant k < l \leqslant n} a_k a_l \cos(k\theta)\cos(l\theta) + \sum_{k=0}^{n} a_k^2 \sin^2(k\theta) \end{aligned}$$

$$
\begin{aligned}
& +2 \sum_{0 \leqslant k<l \leqslant n} a_k a_l \sin(k\theta)\sin(l\theta) \\
= \quad & \sum_{k=0}^{n} a_k^2 + 2 \sum_{0 \leqslant k<l \leqslant n} a_k a_l \cos\big((k-l)\theta\big).
\end{aligned}
$$

On the other hand, we have the equality

$$
\left(\sum_{k=0}^{n} |a_k|\right)^2 = \sum_{k=0}^{n} a_k^2 + 2 \sum_{0 \leqslant k<l \leqslant n} |a_k a_l|.
$$

Then P satisfies $\|P\|_2 = \sum_{k=0}^{n} |a_k|$ if and only if

$$
\sum_{0 \leqslant k<l \leqslant n} a_k a_l \cos\big((k-l)\theta\big) = \sum_{0 \leqslant k<l \leqslant n} |a_k a_l|.
$$

This last assertion means that for all $0 \leqslant k < l \leqslant n$ we have $\cos\big((k-l)\theta\big) = \operatorname{sign}(a_k a_l)$. It is easy to see that this last assertion is equivalent to the assertion 5.

Now, it is trivial that the equality $\|P\|_2 = \sum_{k=0}^{n} |a_k|$ implies the assertions 1 and 2, that 1 implies 3 and that 2 implies 4. Now we show that 4 implies $\|P\|_2 = \sum_{k=0}^{n} |a_k|$. By interpolation, we have

$$
\|P\|_{\infty,S_\infty} = \sum_{k=0}^{n} |a_k| = \|P\|_{p,S^p} \leqslant \big(\|P\|_{\infty,S_\infty}\big)^{1-\frac{2}{p}} \big(\|P\|_{2,S_2}\big)^{\frac{2}{p}}.
$$

Moreover, it is easy to see that $\|P\|_2 = \|P\|_{2,S_2}$. Then we obtain

$$
\big(\|P\|_{\infty,S_\infty}\big)^{\frac{2}{p}} \leqslant \big(\|P\|_{2,S_2}\big)^{\frac{2}{p}} = \big(\|P\|_2\big)^{\frac{2}{p}}.
$$

And finally we have

$$
\sum_{k=0}^{n} |a_k| = \|P\|_{\infty,S_\infty} \leqslant \|P\|_2.
$$

The proof that the assertion 3 implies $\|P\|_2 = \sum_{k=0}^{n} |a_k|$ is similar. ■

Chapter II

Dilation of Ritt operators on L^p-spaces

1 Introduction

Let (Ω, μ) be a measure space and let $1 < p < \infty$. For any bounded operator $T\colon L^p(\Omega) \to L^p(\Omega)$, consider the 'square function'

$$\|x\|_{T,1} = \left\| \Big(\sum_{k=1}^{+\infty} k \big| T^k(x) - T^{k-1}(x) \big|^2 \Big)^{\frac{1}{2}} \right\|_{L^p}, \tag{1.1}$$

defined for any $x \in L^p(\Omega)$. Such quantities frequently appear in the analysis of L^p-operators. They go back at least to [109], where they were used in connection with martingale square functions to study diffusion semigroups and their discrete counterparts. Similar square functions for continuous semigroups played a key role in the recent development of H^∞-calculus and its applications. See in particular the fundamental paper [26], the survey [67] and the references therein.

It is shown in [69] that if T is both a positive contraction and a Ritt operator, then it satisfies a uniform estimate $\|x\|_{T,1} \lesssim \|x\|_{L^p}$ for $x \in L^p(\Omega)$. This estimate and related ones lead to strong maximal inequalities for this class of operators (see also [70]). Next in the paper [68], the author studies the operators T such that both $T\colon L^p(\Omega) \to L^p(\Omega)$ and its adjoint operator $T^*\colon L^{p^*}(\Omega) \to L^{p^*}(\Omega)$ satisfy uniform estimates

$$\|x\|_{T,1} \lesssim \|x\|_{L^p} \qquad \text{and} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}} \tag{1.2}$$

for $x \in L^p(\Omega)$ and $y \in L^{p^*}(\Omega)$. (Here $p^* = \frac{p}{p-1}$ is the conjugate number of p.) It is shown that (1.2) implies that T is an R-Ritt operator (see Section 2 below for the definition) and that (1.2) is equivalent to T having a bounded H^∞-calculus with respect to a Stolz domain of the unit disc with vertex at 1.

The present paper is a continuation of these investigations. Our main result is a characterization of (1.2) in terms of dilations. We show that (1.2) holds true if and only if T is R-Ritt and there exist another measure space $(\widetilde{\Omega}, \widetilde{\mu})$, two bounded maps $J\colon L^p(\Omega) \to L^p(\widetilde{\Omega})$ and $Q\colon L^p(\widetilde{\Omega}) \to L^p(\Omega)$, as well as an isomorphism $U\colon L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ such that $\{U^n : n \in \mathbb{Z}\}$ is bounded and

$$T^n = QU^nJ, \qquad n \geqslant 0.$$

This result will be established in Section 4. It should be regarded as a discrete analog of the main result of [43].

In Section 3, we consider variants of (1.1) as follows. Assume that $T\colon L^p(\Omega) \to L^p(\Omega)$ is a Ritt operator. Then $I - T$ is a sectorial operator and one can define its fractional power $(I-T)^\alpha$ for any $\alpha > 0$. Then we consider

$$\|x\|_{T,\alpha} = \Big\| \Big(\sum_{k=1}^{+\infty} k^{2\alpha-1} \big|T^{k-1}(I-T)^\alpha x\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \tag{1.3}$$

for any $x \in L^p(\Omega)$. Our second main result (Theorem 3.3 below) is that when T is an R-Ritt operator, then the square functions $\| \ \|_{T,\alpha}$ are pairwise equivalent. This result of independent interest should be regarded as a discrete analog of [66, Theorem 1.1]. We prove it here as it is a key step in our characterization of (1.2) in terms of dilations.

Section 2 mostly contains preliminary results. Section 5 is devoted to complements on L^p-operators and their functional calculus properties, in connection with p-completely bounded maps. Finally Section 6 contains generalizations to operators $T\colon X \to X$ on general Banach spaces X. We pay a special attention to noncommutative L^p-spaces, in the spirit of [52].

We end this introduction with a few notation. If X is a Banach space, we let $B(X)$ denote the algebra of all bounded operators on X and we let I_X denote the identity operator on X (or simply I if there is no ambiguity on X). For any $T \in B(X)$, we let $\sigma(T)$ denote the spectrum of T. If $\lambda \in \mathbb{C} \setminus \sigma(T)$ (the resolvent set of T), we let $R(\lambda, T) = (\lambda I_X - T)^{-1}$ denote the corresponding resolvent operator. We refer the reader to [32] for general information on Banach space geometry. We will frequently use Bochner spaces $L^p(\Omega; X)$, for which we refer to [33].

For any $a \in \mathbb{C}$ and $r > 0$, we let $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and we let $\mathbb{D} = D(0, 1)$ denote the open unit disc centered at 0. Also we let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote its boundary.

Whenever $\Omega \subset \mathbb{C}$ is a non empty open set, we let $H^\infty(\Omega)$ denote the space of all bounded holomorphic functions $f\colon \Omega \to \mathbb{C}$. This is a Banach algebra for the norm

$$\|f\|_{H^\infty(\Omega)} = \sup\{|f(z)| : z \in \Omega\}.$$

Also we let $\mathcal{P}$ denote the algebra of all complex polynomials.

In the above presentation and later on in the paper we will use $\lesssim$ to indicate an inequality up to a constant which does not depend on the particular element to which it applies. Then $A(x) \approx B(x)$ will mean that we both have $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

2 Preliminaries on R-boundedness and Ritt operators

This section is devoted to definitions and preliminary results involving R-boundedness (and the companion notion of γ-boundedness), matrix estimates and Ritt operators. We deal with operators acting on an arbitrary Banach space X (as opposed to the next two sections, where X will be an L^p-space).

Let $(\varepsilon_k)_{k\geqslant 1}$ be a sequence of independent Rademacher variables on some probability space Ω_0. We let $\mathrm{Rad}(X) \subset L^2(\Omega_0; X)$ be the closure of $\mathrm{Span}\{\varepsilon_k \otimes x \,:\, k \geqslant 1,\ x \in X\}$ in the Bochner space $L^2(\Omega_0; X)$. Thus for any finite family $x_1, \ldots, x_n$ in X, we have

$$\Bigl\| \sum_{k=1}^{n} \varepsilon_k \otimes x_k \Bigr\|_{\mathrm{Rad}(X)} = \biggl(\int_{\Omega_0} \Bigl\| \sum_{k=1}^{n} \varepsilon_k(\omega)\, x_k \Bigr\|_X^2 \, d\omega \biggr)^{\frac{1}{2}}.$$

We say that a set $F \subset B(X)$ is R-bounded provided that there is a constant $C \geqslant 0$ such that for any finite families $T_1, \ldots, T_n$ in F and $x_1, \ldots, x_n$ in X, we have

$$\Bigl\| \sum_{k=1}^{n} \varepsilon_k \otimes T_k(x_k) \Bigr\|_{\mathrm{Rad}(X)} \leqslant C \Bigl\| \sum_{k=1}^{n} \varepsilon_k \otimes x_k \Bigr\|_{\mathrm{Rad}(X)}.$$

In this case we let $R(F)$ denote the smallest possible C, which is called the R-bound of F.

Let $(g_k)_{k\geqslant 1}$ denote a sequence of independent complex valued, standard Gaussian random variables on some probability space Ω_1, and let $\mathrm{Gauss}(X) \subset L^2(\Omega_1; X)$ be the closure of $\mathrm{Span}\{g_k \otimes x \,:\, k \geqslant 1,\ x \in X\}$. Then replacing the ε_k's and $\mathrm{Rad}(X)$ by the g_k's and $\mathrm{Gauss}(X)$ in the above paragraph, we obtain the similar notion of γ-bounded set. The corresponding γ-bound of a set F is denoted by $\gamma(F)$.

These two notions are very close to each other, however we need to work with both of them in this paper. Comparing them, we recall that any R-bounded set $F \subset B(X)$ is automatically γ-bounded, with $\gamma(F) \leqslant R(F)$. Moreover if X has a finite cotype, then the Rademacher averages and the Gaussian averages are equivalent on X (see e.g. [32, Proposition 12.11 and Theorem. 12.27]), hence F is R-bounded if (and only if) it is γ-bounded.

R-boundedness was introduced in [8] and then developed in the fundamental paper [22]. We refer to the latter paper and to [60, Section 2] for a detailed presentation. We recall two facts which are highly relevant for our paper. First, the closure of the absolute convex hull of any R-bounded set is R-bounded [22, Lemma 3.2]. This implies the following.

Lemma 2.1 *Let $F \subset B(X)$ be an R-bounded set, let $J \subset \mathbb{R}$ be an interval and let $C \geqslant 0$ be a constant. Then the set*

$$\Bigl\{ \int_J a(t) V(t)\, dt \,\Bigm|\, V \colon J \to F \text{ is continuous},\ a \in L^1(J),\ \|a\|_{L^1(J)} \leqslant C \Bigr\}$$

is R-bounded.

Second, if $X = L^p(\Omega)$ is an L^p-space with $1 \leqslant p < \infty$, then X has a finite cotype and we have an equivalence

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(L^p(\Omega))} \approx \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} \tag{2.1}$$

for finite families $(x_k)_k$ of $L^p(\Omega)$. Consequently a set $F \subset B\big(L^p(\Omega)\big)$ is R-bounded if and only if it is γ-bounded, if and only if there exists a constant $C \geqslant 0$ such that for any finite families $T_1, \ldots, T_n$ in F and $x_1, \ldots, x_n$ in $L^p(\Omega)$, we have

$$\Big\| \Big(\sum_{k=1}^n |T_k(x_k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} \leqslant C \Big\| \Big(\sum_{k=1}^n |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)}.$$

In the sequel we represent any element of $B(\ell^2)$ by an infinite matrix $[c_{ij}]_{i,j\geqslant 1}$ in the usual way. Likewise for any integer $n \geqslant 1$, we identify the algebra M_n of all $n \times n$ matrices with the space of linear maps $\ell^2_n \to \ell^2_n$. Clearly an infinite matrix $[c_{ij}]_{i,j\geqslant 1}$ represents an element of $B(\ell^2)$ (in the sense that it is the matrix associated to a bounded operator $\ell^2 \to \ell^2$) if and only if

$$\sup_{n\geqslant 1} \big\| [c_{ij}]_{1\leqslant i,j\leqslant n} \big\|_{B(\ell^2_n)} < \infty\,.$$

For any $[c_{ij}]_{1\leqslant i,j\leqslant n}$ in M_n, we set

$$\big\| [c_{ij}] \big\|_{\mathrm{reg}} = \big\| [|c_{ij}|] \big\|_{B(\ell^2_n)}.$$

This is the so-called 'regular norm' of the operator $[c_{ij}] \colon \ell^2_n \to \ell^2_n$.

Lemma 2.2 *For any matrix $[c_{ij}]$ in M_n, the following assertions are equivalent.*

(i) We have $\big\| [c_{ij}] \big\|_{\mathrm{reg}} \leqslant 1$.

(ii) There exist two matrices $[a_{ij}]$ and $[b_{ij}]$ in M_n such that $c_{ij} = a_{ij} b_{ij}$ for any $i, j = 1, \ldots, n$, and we both have

$$\sup_{1\leqslant i\leqslant n} \sum_{j=1}^n |a_{ij}|^2 \leqslant 1 \qquad \textit{and} \qquad \sup_{1\leqslant j\leqslant n} \sum_{i=1}^n |b_{ij}|^2 \leqslant 1.$$

The implication '(ii)⇒(i)' is an easy application of the Cauchy-Schwarz inequality. The converse is due to Peller [88, Section 3] (see also [2]). We refer to [94] and [102, Section 1.4] for more about this result and complements on regular norms.

The following result extends the boundedness of the Hilbert matrix (wich corresponds to the case $\beta = \gamma = \frac{1}{2}$). We thank Éric Ricard for his precious help in devising this proof.

Proposition 2.3 *Let $\beta, \gamma > 0$ be two positive real numbers. Then the infinite matrix*

$$\left[\frac{i^{\beta - \frac{1}{2}} j^{\gamma - \frac{1}{2}}}{(i+j)^{\beta+\gamma}} \right]_{i,j\geqslant 1}$$

represents an element of $B(\ell^2)$.

Proof : For any $i,j\geqslant 1$, set

$$c_{ij} = \frac{i^{\beta-\frac{1}{2}}j^{\gamma-\frac{1}{2}}}{(i+j)^{\beta+\gamma}},\qquad a_{ij} = c_{ij}^{\frac{1}{2}}\bigl(\tfrac{i}{j}\bigr)^{\frac{1}{4}},\qquad\text{and}\qquad b_{ij} = c_{ij}^{\frac{1}{2}}\bigl(\tfrac{j}{i}\bigr)^{\frac{1}{4}}\,.$$

Then $c_{ij}=a_{ij}b_{ij}$ for any $i,j\geqslant 1$, hence by the easy implication of Lemma 2.2, it suffices to show that

$$\sup_{i\geqslant 1}\sum_{j=1}^{+\infty}|a_{ij}|^2\,<\infty\qquad\text{and}\qquad\sup_{j\geqslant 1}\sum_{i=1}^{+\infty}|b_{ij}|^2\,<\infty\,. \tag{2.2}$$

Fix some $i\geqslant 1$. For any $j\geqslant 1$, we have

$$|a_{ij}|^2 = c_{ij}\bigl(\tfrac{i}{j}\bigr)^{\frac{1}{2}}\,=\,\frac{i^{\beta}j^{\gamma-1}}{(i+j)^{\beta+\gamma}}\,.$$

Hence

$$\sum_{j=1}^{+\infty}|a_{ij}|^2\,=\,i^{\beta}\Biggl(\frac{1}{(i+1)^{\beta+\gamma}}\,+\sum_{j=2}^{+\infty}\frac{1}{j^{1-\gamma}(i+j)^{\beta+\gamma}}\Biggr)\,.$$

Looking at the variations of the function $t\mapsto 1/(t^{1-\gamma}(i+t)^{\beta+\gamma})$ on $(1,\infty)$, we immediately deduce that

$$\sum_{j=1}^{+\infty}|a_{ij}|^2\,\leqslant\,1+2\,i^{\beta}\int_1^{+\infty}\frac{1}{t^{1-\gamma}(i+t)^{\beta+\gamma}}\,dt\,.$$

Changing t into it in the latter integral, we deduce that

$$\sum_{j=1}^{+\infty}|a_{ij}|^2\,\leqslant\,1+2\int_0^{+\infty}\frac{1}{t^{1-\gamma}(1+t)^{\beta+\gamma}}\,dt\,.$$

This upper bound is finite and does not depend on i, which proves the first half of (2.2). The proof of the second half is identical. ∎

We record the following elementary lemma for later use.

Lemma 2.4 *Let* $[c_{ij}]_{i,j\geqslant 1}$ *and* $[d_{ij}]_{i,j\geqslant 1}$ *be infinite matrices of nonnegative real numbers, such that* $c_{ij}\leqslant d_{ij}$ *for any* $i,j\geqslant 1$. *If the matrix* $[d_{ij}]_{i,j\geqslant 1}$ *represents an element of* $B(\ell^2)$, *then the same holds for* $[c_{ij}]_{i,j\geqslant 1}$.

We will need the following classical fact (see e.g. [32, Corollary 12.17]).

Lemma 2.5 *Let* X *be a Banach space and let* $[b_{ij}]_{1\leqslant i,j\leqslant n}$ *be an element of* M_n. *Then for any* $x_1,\ldots,x_n$ *in* X, *we have*

$$\Bigl\|\sum_{i,j=1}^{n}g_i\otimes b_{ij}x_j\Bigr\|_{\mathrm{Gauss}(X)}\,\leqslant\,\bigl\|[b_{ij}]\bigr\|_{B(\ell_n^2)}\Bigl\|\sum_{j=1}^{n}g_j\otimes x_j\Bigr\|_{\mathrm{Gauss}(X)}\,.$$

That result does not remain true if we replace Gaussian variables by Rademacher variables and this defect is the main reason why it is sometimes easier to deal with γ-boundedness than with R-boundedness.

Proposition 2.6 *Let X be a Banach space, let $F = \{T_{ij} \,:\, i, j \geqslant 1\}$ be a γ-bounded family of operators on X, let $n \geqslant 1$ be an integer and let $[c_{ij}]_{1\leqslant i,j\leqslant n}$ be an element of M_n. Then for any $x_1, \ldots, x_n$ in X, we have*

$$\Bigg\| \sum_{i,j=1}^{n} g_i \otimes c_{ij} T_{ij}(x_j) \Bigg\|_{\mathrm{Gauss}(X)} \leqslant \gamma(F) \left\| [c_{ij}] \right\|_{\mathrm{reg}} \Bigg\| \sum_{j=1}^{n} g_j \otimes x_j \Bigg\|_{\mathrm{Gauss}(X)}.$$

Proof : We can assume that $\left\| [c_{ij}] \right\|_{\mathrm{reg}} \leqslant 1$. By Lemma 2.2, we can write $c_{ij} = a_{ij} b_{ij}$ with

$$\sup_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|^2 \leqslant 1 \qquad \text{and} \qquad \sup_{1\leqslant j\leqslant n} \sum_{i=1}^{n} |b_{ij}|^2 \leqslant 1. \tag{2.3}$$

Let $(g_{i,j})_{i,j\geqslant 1}$ be a doubly indexed family of independent Gaussian variables. For any integers $1 \leqslant i, j \leqslant n$, we define

$$A(i) = \begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix} \qquad \text{and} \qquad B(j) = \begin{bmatrix} b_{1j} & b_{2j} & \ldots & b_{nj} \end{bmatrix}^T.$$

Then we consider the two matrices

$$A = \mathrm{Diag}\big(A(1), \ldots, A(n)\big) \in M_{n,n^2} \qquad \text{and} \qquad B = \mathrm{Diag}\big(B(1), \ldots, B(n)\big) \in M_{n^2,n}.$$

Let $x_1, \ldots, x_n \in X$. Applying Lemma 2.5 successively to A and B, we then have

$$\begin{aligned}
\Bigg\| \sum_{i,j=1}^{n} g_i \otimes c_{ij} T_{ij}(x_j) \Bigg\|_{\mathrm{Gauss}(X)} &= \Bigg\| \sum_{i,j=1}^{n} g_i \otimes a_{ij} b_{ij} T_{ij}(x_j) \Bigg\|_{\mathrm{Gauss}(X)} \\
&\leqslant \|A\| \Bigg\| \sum_{i,j=1}^{n} g_{ij} \otimes b_{ij} T_{ij}(x_j) \Bigg\|_{\mathrm{Gauss}(X)} \\
&\leqslant \gamma(F) \, \|A\| \Bigg\| \sum_{i,j=1}^{n} g_{ij} \otimes b_{ij} x_j \Bigg\|_{\mathrm{Gauss}(X)} \\
&\leqslant \gamma(F) \, \|A\| \|B\| \Bigg\| \sum_{j=1}^{n} g_j \otimes x_j \Bigg\|_{\mathrm{Gauss}(X)}.
\end{aligned}$$

We have

$$\|A\| = \sup_{1\leqslant i\leqslant n} \left\| A(i) \right\|_{M_{1,n}} = \sup_{1\leqslant i\leqslant n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

hence $\|A\| \leqslant 1$ by (2.3). Likewise, we have $\|B\| \leqslant 1$ hence the above inequality yields the result. ■

We now turn to Ritt operators, the key class of this paper, and recall some of their main features. Details and complements can be found in [15, 16, 68, 73, 77, 80, 116]. We say that an operator $T \in B(X)$ is a Ritt operator if the two sets

$$\{T^n \,:\, n \geqslant \} \qquad \text{and} \qquad \Big\{n(T^n - T^{n-1}) \,:\, n \geqslant 1\Big\} \tag{2.4}$$

are bounded. This is equivalent to the spectral inclusion

$$\sigma(T) \subset \overline{\mathbb{D}} \tag{2.5}$$

and the boundedness of the set

$$\Big\{(\lambda - 1)R(\lambda, T) \,:\, |\lambda| > 1\Big\}. \tag{2.6}$$

This resolvent estimate outside the unit disc is called the 'Ritt condition'.

Likewise we say that T is an R-Ritt operator if the two sets in (2.4) are R-bounded. This is equivalent to the inclusion (2.5) and the R-boundedness of the set (2.6).

For any angle $\gamma \in (0, \frac{\pi}{2})$, let B_γ be the interior of the convex hull of 1 and the disc $D(0, \sin\gamma)$ (see Figure II.1 below).

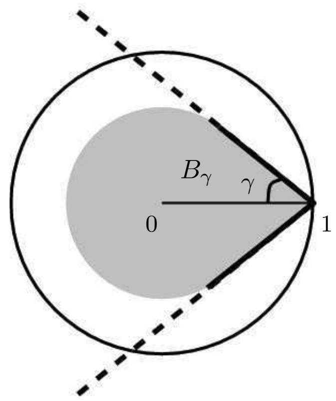

Figure II.1

Then the Ritt condition and its R-bounded version can be strengthened as follows.

Lemma 2.7 *Let $T\colon X \to X$ be a Ritt operator (resp. an R-Ritt operator). There exists an angle $\gamma \in (0, \frac{\pi}{2})$ such that*

$$\sigma(T) \subset B_\gamma \cup \{1\} \tag{2.7}$$

and the set

$$\Big\{(\lambda - 1)R(\lambda, T) \,:\, \lambda \in \mathbb{C} \setminus B_\gamma,\ \lambda \neq 1\Big\} \tag{2.8}$$

is bounded (resp. R-bounded).

This essentially goes back to [15], see [68] for details.

For any angle $\theta \in (0, \pi)$, let

$$\Sigma_\theta = \Big\{ z \in \mathbb{C} \,:\, |\mathrm{Arg}(z)| < \theta \Big\} \tag{2.9}$$

be the open sector of angle 2θ around the positive real axis $(0, \infty)$. We say that a closed operator $A\colon D(A) \to X$ with dense domain $D(A)$ is sectorial if there exists $\theta \in (0, \pi)$ such that $\sigma(A) \subset \overline{\Sigma_\theta}$ and the set

$$\Big\{ zR(z, A) \,:\, z \in \mathbb{C} \setminus \overline{\Sigma_\theta} \Big\} \tag{2.10}$$

is bounded.

Let T be a Ritt operator and let $\gamma \in (0, \frac{\pi}{2})$ be such that the spectral inclusion (2.7) holds true and the set (2.8) is bounded. Then $A = I - T$ is a sectorial operator. Indeed $1 - B_\gamma \subset \Sigma_\gamma$ and $zR(z, A) = ((1 - z) - 1)R(1 - z, T)$ for any $z \notin \overline{\Sigma_\gamma}$. Hence for $\theta = \gamma$, the set (2.10) is bounded. Thus for any $\alpha > 0$, one can consider the fractional power $(I - T)^\alpha$. We refer e.g. to [45, Chapter 3] for various definitions of these (bounded) operators and their basic properties. Fractional powers of Ritt operators can be expressed by a natural Dunford-Riesz functional calculus formula. Indeed it was observed in [68] that for any polynomial φ, we have

$$\varphi(T)(I - T)^\alpha \;=\; \frac{1}{2\pi i} \int_{\partial B_\gamma} \varphi(\lambda)(1 - \lambda)^\alpha R(\lambda, T)\, d\lambda\,, \tag{2.11}$$

where the countour ∂B_γ is oriented counterclockwise.

P. Vitse proved in [116] that if $T\colon X \to X$ is a Ritt operator, then for any integer $N \geqslant 0$, the set $\{n^N T^{n-1}(I - T)^N \,:\, n \geqslant 1\}$ is bounded. Our next statement is a continuation of these results.

Proposition 2.8 *Let X be a Banach space and let $T\colon X \to X$ be a Ritt operator (resp. an R-Ritt operator). For any $\alpha > 0$, the set*

$$\Big\{ n^\alpha (rT)^{n-1}(I - rT)^\alpha \,:\, n \geqslant 1,\; r \in (0, 1] \Big\}$$

is bounded (resp. R-bounded).

Proof : We will prove this result in the 'R-Ritt case' only. The 'Ritt case' is similar and simpler. Assume that T is R-Ritt. Applying Lemma 2.7, we let $\gamma \in (0, \frac{\pi}{2})$ be such that (2.7) holds true and the set (2.8) is R-bounded. Let $r \in (0, 1]$ and let $\lambda \in \mathbb{C} \setminus B_\gamma$, with $\lambda \neq 1$. Then $\frac{\lambda}{r} \in \mathbb{C} \setminus B_\gamma$ hence $\frac{\lambda}{r}$ belongs to the resolvent set of T and we have

$$(\lambda - 1)R(\lambda, rT) = \frac{\lambda - 1}{\lambda - r}\Big(\frac{\lambda}{r} - 1\Big) R\Big(\frac{\lambda}{r}, T\Big).$$

Since the set

$$\left\{\frac{\lambda-1}{\lambda-r}\ :\ \lambda\in\mathbb{C}\setminus B_\gamma,\ \lambda\neq 1,\ r\in(0,1]\right\}$$

is bounded, it follows from the above formula that the set

$$\Big\{(\lambda-1)R(\lambda,rT)\ :\ \lambda\in\mathbb{C}\setminus B_\gamma,\ \lambda\neq 1,\ r\in(0,1]\Big\} \tag{2.12}$$

is R-bounded.

The boundary ∂B_γ is the juxtaposition of the segment Γ_+ going from 1 to $1-\cos(\gamma)e^{-i\gamma}$, of the segment Γ_- going from $1-\cos(\gamma)e^{i\gamma}$ to 1 and of the curve Γ_0 going from $1-\cos(\gamma)e^{-i\gamma}$ to $1-\cos(\gamma)e^{i\gamma}$ counterclockwise along the circle of center 0 and radius $\sin\gamma$.

Consider a fixed number $\alpha>0$. For any integer $n\geqslant 1$ and any $r\in(0,1]$, we have

$$(rT)^{n-1}(I-rT)^\alpha\ =\ \frac{1}{2\pi i}\int_{\partial B_\gamma}\lambda^{n-1}(1-\lambda)^\alpha R(\lambda,rT)\,d\lambda$$

by applying (2.11) to rT. Hence we may write

$$n^\alpha(rT)^{n-1}(I-rT)^\alpha\ =\ \frac{-n^\alpha}{2\pi i}\int_{\partial B_\gamma}\lambda^{n-1}(1-\lambda)^{\alpha-1}(\lambda-1)R(\lambda,rT)\,d\lambda\,.$$

According to the R-boundedness of the set (2.12) and Lemma 2.1, it therefore suffices to show that the integrals

$$I_n\ =\ n^\alpha\int_{\partial B_\gamma}|\lambda|^n|1-\lambda|^{\alpha-1}\,|d\lambda|$$

are uniformly bounded (for n varying in $\mathbb{N}$). Let us decompose each of these integrals as $I_n = I_{n,0}+I_{n,+}+I_{n,-}$, with

$$I_{n,0}=n^\alpha\int_{\Gamma_0}\cdots\,|d\lambda|\,,\quad I_{n,+}=n^\alpha\int_{\Gamma_+}\cdots\,|d\lambda|\,,\quad\text{and}\quad I_{n,-}=n^\alpha\int_{\Gamma_-}\cdots\,|d\lambda|\,.$$

For $\lambda\in\Gamma_0$, we both have

$$\cos\gamma\leqslant|1-\lambda|\leqslant 2\qquad\text{and}\qquad|\lambda|=\sin\gamma.$$

Since the sequence $\big(n^\alpha(\sin\gamma)^n\big)_{n\geqslant 1}$ is bounded, this readily implies that the sequence $(I_{n,0})_{n\geqslant 1}$ is bounded.

Let us now estimate $I_{n,+}$. For any $t\in[0,\cos\gamma]$, we have $t^2\leqslant t\cos\gamma$ hence

$$\left|1-te^{-i\gamma}\right|^2\ =\ 1+t^2-2t\cos\gamma\leqslant 1-t\cos\gamma.$$

Hence

$$I_{n,+}=n^\alpha\int_0^{\cos\gamma}\left|1-te^{-i\gamma}\right|^n t^{\alpha-1}\,dt\ \leqslant n^\alpha\int_0^{\cos\gamma}(1-t\cos\gamma)^{\frac{n}{2}}t^{\alpha-1}\,dt\,.$$

Changing t into $s = t\cos\gamma$ and using the inequality $1 - s \leqslant e^{-s}$, we deduce that

$$I_{n,+} \leqslant \frac{n^\alpha}{(\cos\gamma)^\alpha}\int_0^{\cos^2\gamma} s^{\alpha-1}\, e^{-\frac{sn}{2}}\, ds\,.$$

This yields (changing s into $u = \frac{sn}{2}$)

$$I_{n,+} \leqslant \frac{2^\alpha}{(\cos\gamma)^\alpha}\int_0^{+\infty} u^{\alpha-1}e^{-u}\, du\,.$$

Thus the sequence $(I_{n,+})_{n\geqslant 1}$ is bounded. Since $I_{n,-} = I_{n,+}$, this completes the proof of the boundedness of $(I_n)_{n\geqslant 1}$. ■

3 Equivalence of square functions

Throughout the next two sections, we fix a measure space (Ω, μ) and a number $1 < p < \infty$. We shall deal with operators acting on the Banach space $X = L^p(\Omega)$. We start with a precise definition of (1.1) and (1.3) and a few comments.

Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a bounded operator and let $x \in L^p(\Omega)$. Let us consider

$$x_k = k^{\frac{1}{2}}\big(T^k(x) - T^{k-1}(x)\big)$$

for any $k \geqslant 1$. If the sequence $(x_k)_{k\geqslant 1}$ belongs to the space $L^p(\Omega; \ell^2)$, then $\|x\|_{T,1}$ is defined as the norm of $(x_k)_{k\geqslant 1}$ in that space. Otherwise, we set $\|x\|_{T,1} = \infty$. If T is a Ritt operator, then the quantities $\|x\|_{T,\alpha}$ are defined in a similar manner for any $\alpha > 0$. In particular, $\|x\|_{T,\alpha}$ can be infinite.

These square functions are natural discrete analogs of the square functions asociated to sectorial operators (see [26] and the survey paper [67]).

Assume that T is a Ritt operator. Then T is power bounded hence by the Mean Ergodic Theorem (see e.g. [59, Section 2.1]), we have a direct sum decomposition

$$L^p(\Omega) \;=\; \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Ran}(I - T)}, \tag{3.1}$$

where $\mathrm{Ker}(\cdot\,)$ and $\mathrm{Ran}(\cdot\,)$ denote the kernel and the range, respectively. For any $\alpha > 0$, we have $\mathrm{Ker}\big((I - T)^\alpha\big) = \mathrm{Ker}(I - T)$. This implies that

$$\|x\|_{T,\alpha} = 0 \iff x \in \mathrm{Ker}(I - T). \tag{3.2}$$

Given any $\alpha > 0$, a general question is to determine whether $\|x\|_{T,\alpha} < \infty$ for any x in $L^p(\Omega)$. It is easy to check, using the Closed graph Theorem, that this finiteness property is equivalent to the existence of a constant $C \geqslant 0$ such that

$$\|x\|_{T,\alpha} \leqslant C\|x\|_{L^p}, \qquad x \in L^p(\Omega). \tag{3.3}$$

In [68], C. Le Merdy established the following connection between the boundedness of discrete square functions and functional calculus properties.

Theorem 3.1 *([68]) Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. The following assertions are equivalent.*

(i) The operator T and its adjoint $T^*\colon L^{p^*}(\Omega) \to L^{p^*}(\Omega)$ both satisfy uniform estimates

$$\|x\|_{T,1} \lesssim \|x\|_{L^p} \qquad \textit{and} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}}$$

for $x \in L^p(\Omega)$ and $y \in L^{p^*}(\Omega)$.

(ii) There exists an angle $0 < \gamma < \frac{\pi}{2}$ and a constant $K \geqslant 0$ such that

$$\|\varphi(T)\| \leqslant K\, \|\varphi\|_{H^\infty(B_\gamma)}$$

for any $\varphi \in \mathcal{P}$.

(iii) The operator T is R-Ritt and there exists an angle $0 < \theta < \pi$ such that $I - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.

Besides [68], we refer to [26, 60, 65, 75] for general information on $H^\infty(\Sigma_\theta)$ functional calculus for sectorial operators.

The main purpose of this section is to show that if T is R-Ritt, then the square functions $\| \ \|_{T,\alpha}$ are pairwise equivalent. Thus the existence of an estimate (3.3) does not depend on $\alpha > 0$. This result (Theorem 3.3 below) is a discrete analog of the equivalence of square functions associated to R-sectorial operators, as established in [66].

We start with preliminary results which allow to estimate square functions $\|x\|_{T,\alpha}$ by means of approximation processes.

Lemma 3.2 *Assume that $T\colon L^p(\Omega) \to L^p(\Omega)$ is a Ritt operator, and let $\alpha > 0$.*

(1) For any operator $V\colon L^p(\Omega) \to L^p(\Omega)$ such that $VT = TV$ and any $x \in L^p(\Omega)$, we have

$$\big\|V(x)\big\|_{T,\alpha} \leqslant \|V\|\|x\|_{T,\alpha}.$$

(2) For any $x \in \mathrm{Ran}(I - T)$, we have $\|x\|_{T,\alpha} < \infty$.

(3) Let $\nu \geqslant \alpha + 1$ be an integer and let $x \in \mathrm{Ran}\big((I - T)^\nu\big)$. Then

$$\|x\|_{T,\alpha} \;=\; \lim_{r \to 1^-} \|x\|_{rT,\alpha}.$$

Proof : (1): Consider $V \in B\big(L^p(\Omega)\big)$. As is well-known, the tensor product $V \otimes I_{\ell^2}$ extends to a bounded operator $V\overline{\otimes}I_{\ell^2} \colon L^p(\Omega;\ell^2) \longrightarrow L^p(\Omega;\ell^2)$, with $\|V\overline{\otimes}I_{\ell^2}\| = \|V\|$. Assume that $VT = TV$ and let x be such that $\|x\|_{T,\alpha} < \infty$. Then we have

$$\Big(k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^{\alpha}\big(V(x)\big)\Big)_{k\geqslant 1} \;=\; V\overline{\otimes}I_{\ell^2}\,\Big[\big(k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^{\alpha}(x)\big)_{k\geqslant 1}\Big],$$

and the result follows at once.

(2): Assume that $x = (I-T)x'$ for some $x' \in L^p(\Omega)$. By Proposition 2.8, there exists a constant C such that

$$\begin{aligned}
\sum_{k=1}^{\infty}\Big\|k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^{\alpha}(x)\Big\|_{L^p} \;&=\; \sum_{k=1}^{\infty}k^{\alpha-\frac{1}{2}}\Big\|T^{k-1}(I-T)^{\alpha+1}(x')\Big\|_{L^p}\\
&\leqslant\; \|x'\|_{L^p}\sum_{k=1}^{\infty}k^{\alpha-\frac{1}{2}}\,\frac{C}{k^{\alpha+1}}\\
&\leqslant\; C\,\|x'\|_{L^p}\sum_{k=1}^{\infty}k^{-\frac{3}{2}} \;<\infty.
\end{aligned}$$

This implies that $\big(k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^{\alpha}(x)\big)_{k\geqslant 1}$ belongs to $L^p(\Omega;\ell^2)$.

(3): It is clear that $(I-rT)^{\alpha} \to (I-T)^{\alpha}$ when $r \to 1^-$. Assume that $x \in \mathrm{Ran}\big((I-T)^{\nu}\big)$. Arguing as in part (2) we find that the sequence $\big(k^{\alpha-\frac{1}{2}}T^{k-1}(x)\big)_{k\geqslant 1}$ belongs to $L^p(\Omega;\ell^2)$. Then arguing as in part (1), we obtain that

$$\Big\|\Big(k^{\alpha-\frac{1}{2}}T^{k-1}\big((I-rT)^{\alpha}-(I-T)^{\alpha}\big)(x)\Big)_{k\geqslant 1}\Big\|_{L^p(\ell^2)} \longrightarrow 0$$

when $r \to 1^-$. This implies the convergence result. ∎

Theorem 3.3 *Assume that $T\colon L^p(\Omega) \to L^p(\Omega)$ is an R-Ritt operator. Then for any $\alpha,\beta > 0$, we have an equivalence*

$$\|x\|_{T,\alpha} \;\approx\; \|x\|_{T,\beta}, \qquad x \in L^p(\Omega).$$

Proof : We fix $\gamma > 0$ such that $\alpha+\gamma$ is an integer $N \geqslant 1$. For any integer $k \geqslant 1$, we define the complex number

$$c_k = \frac{k(k+1)\cdots(k+N-2)}{k^{\alpha-\frac{1}{2}}},$$

with the convention that $c_k = \frac{1}{k^{\alpha-\frac{1}{2}}}$ if $N=1$. For any $z \in \mathbb{D}$, we have

$$\sum_{k=1}^{\infty}k(k+1)\cdots(k+N-2)z^{k-1} \;=\; \frac{(N-1)!}{(1-z)^N}.$$

Hence

$$\sum_{k=1}^{\infty} c_k k^{\alpha-\frac{1}{2}} z^{2k-2}(1-z^2)^N \;=\; \sum_{k=1}^{\infty} k(k+1)\cdots(k+N-2)(z^2)^{k-1}(1-z^2)^N \;=\; (N-1)!.$$

Since the operator T is power bounded, we deduce that for every $r \in (0,1)$ we have

$$\sum_{k=1}^{\infty} c_k k^{\alpha-\frac{1}{2}} (rT)^{2k-2}\big(I-(rT)^2\big)^N \;=\; (N-1)!I,$$

the series being absolutely convergent. Since $(I+rT)^N$ is invertible, this yields

$$\sum_{k=1}^{\infty} c_k (rT)^{k-1}(I-rT)^{\gamma} k^{\alpha-\frac{1}{2}} (rT)^{k-1}(I-rT)^{\alpha} = (N-1)!(I+rT)^{-N}.$$

Let $x \in L^p(\Omega)$. For any integer $m \geqslant 1$ and any $r \in (0,1)$, we let

$$y_m(r) \;=\; (N-1)!(I+rT)^{-N} m^{\beta-\frac{1}{2}} (rT)^{m-1}(I-rT)^{\beta} x.$$

Then it follows from the above identity that

$$y_m(r) \;=\; \sum_{k=1}^{\infty} c_k m^{\beta-\frac{1}{2}} (rT)^{m+k-2}(I-rT)^{\beta+\gamma} \cdot k^{\alpha-\frac{1}{2}} (rT)^{k-1}(I-rT)^{\alpha} x.$$

For any $n \geqslant 1$, we consider the partial sum

$$y_{m,n}(r) \;=\; \sum_{k=1}^{n} c_k m^{\beta-\frac{1}{2}} (rT)^{m+k-2}(I-rT)^{\beta+\gamma} \cdot k^{\alpha-\frac{1}{2}} (rT)^{k-1}(I-rT)^{\alpha} x,$$

and we have $y_{m,n}(r) \to y_m(r)$ when $n \to \infty$. Let us write

$$c_k m^{\beta-\frac{1}{2}} (rT)^{m+k-2}(I-rT)^{\beta+\gamma} \;=\; \frac{m^{\beta-\frac{1}{2}} c_k}{(m+k-1)^{\gamma+\beta}} \Big[(m+k-1)^{\gamma+\beta}(rT)^{m+k-2}(I-rT)^{\beta+\gamma}\Big] \tag{3.4}$$

for any $m,k \geqslant 1$. Since $c_k \sim_{+\infty} k^{\gamma-\frac{1}{2}}$, there exists a positive constant K such that

$$\frac{m^{\beta-\frac{1}{2}} c_k}{(m+k-1)^{\gamma+\beta}} \leqslant K\, \frac{m^{\beta-\frac{1}{2}} k^{\gamma-\frac{1}{2}}}{(m+k)^{\gamma+\beta}}$$

for any $m,k \geqslant 1$. It therefore follows from Proposition 2.3 and Lemma 2.4 that the matrix

$$\left[\frac{m^{\beta-\frac{1}{2}} c_k}{(m+k-1)^{\gamma+\beta}}\right]_{m,k\geqslant 1}$$

represents an element of $B(\ell^2)$. Moreover, by Proposition 2.8, the set

$$F \,=\, \Big\{(m+k-1)^{\gamma+\beta}(rT)^{m+k-2}(I-rT)^{\gamma+\beta} \,:\, m,k\geqslant 1,\ r\in(0,1]\Big\}$$

is R-bounded. Hence by (2.1), (3.4) and Proposition 2.6, we get to an estimate

$$\Big\|\Big(\sum_{m=1}^{M}\big|y_{m,n}(r)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \lesssim \Big\|\Big(\sum_{k=1}^{+\infty}k^{2\alpha-1}\big|(rT)^{k-1}(I-rT)^{\alpha}x\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}$$

for any integer $M\geqslant 1$. Passing to the limit, we deduce that

$$\Big\|\Big(\sum_{m=1}^{\infty}\big|y_{m}(r)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \lesssim \Big\|\Big(\sum_{k=1}^{+\infty}k^{2\alpha-1}\big|(rT)^{k-1}(I-rT)^{\alpha}x\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}.$$

Since the set $\big\{(N-1)!^{-1}(I+rT)^N \,:\, r\in(0,1)\big\}$ is bounded, we finally obtain that

$$\|x\|_{rT,\beta} \lesssim \|x\|_{rT,\alpha}.$$

It is crucial to note that in this estimate, the majorizing constant hidden in the symbol $\lesssim$ does not depend on $r\in(0,1)$. Now let ν be an integer such that $\nu\geqslant\alpha+1$ and $\nu\geqslant\beta+1$. Applying Lemma 3.2 (3), we deduce a uniform estimate

$$\|x\|_{T,\beta}\lesssim\|x\|_{T,\alpha}$$

for $x\in\mathrm{Ran}\big((I-T)^{\nu}\big)$. Next for any integer $m\geqslant 0$, set

$$\Lambda_m=\frac{1}{m+1}\sum_{k=0}^{m}(I-T^k).$$

It is clear that Λ_m^{ν} maps X into $\mathrm{Ran}\big((I-T)^{\nu}\big)$. Hence we actually have a uniform estimate

$$\big\|\Lambda_m^{\nu}(x)\big\|_{T,\beta} \lesssim \big\|\Lambda_m^{\nu}(x)\big\|_{T,\alpha}, \qquad x\in X,\ m\geqslant 1.$$

Since T is power bounded, the sequence $(\Lambda_m)_{m\geqslant 0}$ is bounded. Applying Lemma 3.2 (1), we deduce a further uniform estimate

$$\big\|\Lambda_m^{\nu}(x)\big\|_{T,\beta} \lesssim \|x\|_{T,\alpha}, \qquad x\in X,\ m\geqslant 1.$$

Equivalently, we have

$$\Big\|\Big(\sum_{k=1}^{l}k^{2\beta-1}\big|T^{k-1}(I-T)^{\beta}\Lambda_m^{\nu}(x)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \lesssim \|x\|_{T,\alpha}, \qquad x\in X,\ m\geqslant 1,\ l\geqslant 1.$$

For any $x\in\overline{\mathrm{Ran}(I-T)}$, $\Lambda_m(x)\to x$ and hence $\Lambda_m^{\nu}(x)\to x$ when $m\to\infty$. Hence passing to the limit in the above inequaliy, we obtain a uniform estimate $\|x\|_{T,\beta} \lesssim \|x\|_{T,\alpha}$ for x in $\overline{\mathrm{Ran}(I-T)}$. Switching the roles of α and β, this shows that $\| \ \|_{T,\beta}$ and $\| \ \|_{T,\alpha}$ are equivalent on the space $\overline{\mathrm{Ran}(I-T)}$. Moreover $\| \ \|_{T,\beta}$ and $\| \ \|_{T,\alpha}$ vanish on $\mathrm{Ker}(I-T)$ by (3.2). Appealing to the direct sum decomposition (3.1), we finally obtain that $\| \ \|_{T,\beta}$ and $\| \ \|_{T,\alpha}$ are equivalent on $L^p(\Omega)$. ∎

The techniques developed so far in this paper allow us to prove the following proposition, which complements Theorem 3.1. For a Ritt operator T, we let P_T denote the projection onto $\mathrm{Ker}(I-T)$ which vanishes on $\overline{\mathrm{Ran}(I-T)}$ (recall (3.1)).

Proposition 3.4 *Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. Then the condition (i) in Theorem 3.1 is equivalent to:*

(i)' We have an equivalence

$$\|x\|_{L^p} \approx \big\|P_T(x)\big\|_{L^p} + \|x\|_{T,1}$$

for $x \in L^p(\Omega)$.

Proof : That (i) implies (i)' was proved in [69, Remark 3.4] in the case when T is 'contractively regular'. The proof in our present case is the same.

Assume (i)'. Let $y \in L^{p^*}(\Omega)$. We consider a finite sequence $(x_k)_{k\geqslant 1}$ in $L^p(\Omega)$ and we set

$$x = \sum_k k^{\frac{1}{2}} T^{k-1}(I-T)x_k\,.$$

Then

$$\Big|\sum_k \langle k^{\frac{1}{2}}(T^*)^{k-1}(I-T^*)y, x_k\rangle\Big| = |\langle y, x\rangle| \leqslant \|x\|_{L^p}\|y\|_{L^{p^*}}.$$

Moreover $x \in \mathrm{Ran}(I-T)$ hence applying (i)', we deduce

$$\Big|\sum_k \langle k^{\frac{1}{2}}(T^*)^{k-1}(I-T^*)y, x_k\rangle\Big| \lesssim \|y\|_{L^{p^*}}\|x\|_{T,1}.$$

We will now show an estimate

$$\|x\|_{T,1} \lesssim \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{3.5}$$

Then passing to the supremum over all finite sequences $(x_k)_{k\geqslant 1}$ in the unit ball of $L^p(\Omega;\ell^2)$, we deduce that $\|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}}$.

To show (3.5), first note that for any integer $m \geqslant 1$, we may write

$$m^{\frac{1}{2}} T^{m-1}(I-T)x = \sum_k \frac{m^{\frac{1}{2}} k^{\frac{1}{2}}}{(m+k)^2}\,(m+k)^2 T^{m+k-2}(I-T)^2 x_k\,.$$

Second according to [68], the assumption (i)' implies that T is an R-Ritt operator. Hence by Proposition 2.8, the set $\{(m+k)^2 T^{m+k-2}(I-T)^2 \,:\, m,k \geqslant 1\}$ is R-bounded. Therefore applying Propositions 2.3 and 2.6 we obtain (3.5). ■

4 Loose dilations

We will focus on the following notion of dilation for L^p-operators.

Definition 4.1 *Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a bounded operator. We say that it admits a loose dilation if there exist a measure space $(\widetilde{\Omega}, \widetilde{\mu})$, two bounded maps $J\colon L^p(\Omega) \to L^p(\widetilde{\Omega})$ and $Q\colon L^p(\widetilde{\Omega}) \to L^p(\Omega)$, as well as an isomorphism $U\colon L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ such that $\{U^n \,:\, n \in \mathbb{Z}\}$ is bounded and*

$$T^n = QU^nJ, \qquad n \geqslant 0.$$

That notion is strictly weaker than the following more classical one.

Remark 4.2 *We say that a bounded operator $T\colon L^p(\Omega) \to L^p(\Omega)$ admits a strict dilation if there exist a measure space $(\widetilde{\Omega}, \widetilde{\mu})$, two contractions $J\colon L^p(\Omega) \to L^p(\widetilde{\Omega})$ and $Q\colon L^p(\widetilde{\Omega}) \to L^p(\Omega)$, as well as an isometric isomorphism $U\colon L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ such that $T^n = QU^nJ$ for any $n \geqslant 0$.*

This strict dilation property implies that T is a contraction and that J and Q^* are both isometries.

Conversely in the case $p = 2$, Nagy's dilation Theorem (see e.g. [113, Chapter 1]) ensures that any contraction $L^2(\Omega) \to L^2(\Omega)$ admits a strict dilation.

Next, assume that $1 < p \neq 2 < \infty$. Then it follows from [2, 3, 23, 88] that $T\colon L^p(\Omega) \to L^p(\Omega)$ admits a strict dilation if and only if there exists a positive contraction $S\colon L^p(\Omega) \to L^p(\Omega)$ such that $|T(x)| \leqslant S(|x|)$ for any $x \in L^p(\Omega)$.

Except for $p = 2$ (see Remark 4.3 below), there is no similar description of operators admitting a loose dilation. The general issue behind our investigation is to try to characterize the L^p-operators which satisfy this property. Theorem 4.8 below gives a satisfactory answer for the class of Ritt operators.

Remark 4.3 *Let H be a Hilbert space, let $T\colon H \to H$ be a bounded operator and let us say that T admits a loose dilation if there exist a Hilbert space K, two bounded maps $J\colon H \to K$ and $Q\colon K \to H$, and an isomorphism $U\colon K \to K$ such that $\{U^n \,:\, n \in \mathbb{Z}\}$ is bounded and $T^n = QU^nJ$ for any $n \geqslant 0$. Then this property is equivalent to T being similar to a contraction.*

Indeed assume that there exists an isomorphism $V \in B(H)$ such that $V^{-1}TV$ is a contraction. By Nagy's dilation Theorem, that contraction admits a unitary dilation. In other words, there is a unitary U on a Hilbert space K containing H, such that $(V^{-1}TV)^n = qU^nj$ for any $n \geqslant 0$, where $j\colon H \to K$ is the canonical inclusion and $q = j^*$ is the corresponding orthogonal projection. We obtain the loose dilation property of T by taking $J = jV^{-1}$ and $Q = Vq$.

The converse uses the notion of complete polynomial boundedness, for which we refer to [86, 87]. Assume that T admits a loose dilation. Using [86, Corollary 9.4] and elementary arguments, we obtain that T is completely polynomially bounded. Hence it is similar to a contraction by [87, Corollary 3.5].

According to the above result, the rest of this section is significant only in the case $1 < p \neq 2 < \infty$.

Let $S\colon \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}$ denote the natural shift operator given by $S\big((t_k)_{k\in\mathbb{Z}}\big) = \big((t_{k-1})_{k\in\mathbb{Z}}\big)$. For any φ in $\mathcal{P}$ (the algebra of complex polynomials), we set

$$\|\varphi\|_p = \big\|\varphi(S)\big\|_{B(\ell^p_{\mathbb{Z}})}. \tag{4.1}$$

We recall that if φ is given by $\varphi(z) = \sum_{k\geqslant 0} d_k z^k$, then $\varphi(S)$ is the convolution operator (with respect to the group $\mathbb{Z}$) associated to the sequence $(d_k)_{k\in\mathbb{Z}}$. Alternatively, $\varphi(S)$ is the Fourier multiplier associated to the restriction of φ to the unitary group $\mathbb{T}$. We refer the reader to [35] for some elementary background on Fourier multiplier theory.

Let us decompose $(0, \pi)$ dyadically into the following family $(I_j)_{j\in\mathbb{Z}}$ of intervals:

$$I_j = \begin{cases} \big[\pi - \frac{\pi}{2^{j+1}}, \pi - \frac{\pi}{2^{j+2}}\big) & \text{if } j \geqslant 0 \\ \big[2^{j-1}\pi, 2^j\pi\big) & \text{if } j < 0. \end{cases}$$

Then we denote by Δ_j the corresponding arcs of $\mathbb{T}$:

$$\Delta_j = \Big\{e^{it} \ : \ t \in -I_j \cup I_j \ \Big\}.$$

We will use the following version of the Marcinkiewicz multiplier theorem (see [15, Theorem 4.3] and also [35]).

Theorem 4.4 *Let $1 < p < \infty$. Let $\phi \in L^\infty(\mathbb{T})$ and assume that ϕ has uniformly bounded variations over the $(\Delta_j)_{j\in\mathbb{Z}}$. Then ϕ induces a bounded Fourier multiplier $M_\phi\colon \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}$ and we have*

$$\|M_\phi\|_{B(\ell^p_{\mathbb{Z}})} \leqslant C_p \Big(\|\phi\|_{L^\infty(\mathbb{T})} + \sup\{\mathrm{var}\,(\phi, \Delta_j) \ : \ j \in \mathbb{Z}\}\Big),$$

where $\mathrm{var}\,(\phi, \Delta_j)$ *is the usual variation of ϕ over Δ_j and the constant C_p only depends on p.*

For convenience, Definition 4.5 and Proposition 4.7 below are given for an arbitrary Banach space X, although we are mostly interested in the case when X is an L^p-space.

Definition 4.5 *We say that a bounded operator $T\colon X \to X$ is p-polynomially bounded is there exists a constant $C \geqslant 1$ such that*

$$\|\varphi(T)\| \leqslant C\|\varphi\|_p \tag{4.2}$$

for any complex polynomial φ.

The following connection with dilations is well-known.

Proposition 4.6 *If $T\colon L^p(\Omega) \to L^p(\Omega)$ admits a loose dilation, then it is p-polynomially bounded.*

Proof : Assume that T satisfies the dilation property given by Definition 4.1. Then for any $\varphi \in \mathcal{P}$, we have $\varphi(T) = Q\varphi(U)J$, hence

$$\|\varphi(T)\| \leqslant \|Q\|\|J\|\|\varphi(U)\|.$$

Moreover by the transference principle (see [24, Theorem 2.4]), $\|\varphi(U)\| \leqslant K^2\|\varphi\|_p$, where $K \geqslant 1$ is any constant such that $\|U^n\| \leqslant K$ for any integer n. This yields the result. ■

We will see in Section 5 that the converse of that proposition does not hold true.

The above proof shows that if $T\colon L^p(\Omega) \to L^p(\Omega)$ admits a strict dilation, then $\|\varphi(T)\| \leqslant \|\varphi\|_p$ for any $\varphi \in \mathcal{P}$, a very classical fact. The famous Matsaev Conjecture asks whether this inequality holds for any L^p-contraction T (even those with no strict dilation). This was disproved very recently by Drury in the case $p = 4$ [34]. It is unclear whether there exists an L^p-contraction T satisfying $\|\varphi(T)\| \leqslant \|\varphi\|_p$ for any $\varphi \in \mathcal{P}$, without admitting a strict dilation.

Proposition 4.7 *Let $T\colon X \to X$ be a p-polynomially bounded operator. Then $I - T$ is sectorial and for any $\theta \in \left(\frac{\pi}{2}, \pi\right)$, it admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof : Since T is p-polynomially bounded, it is power bounded hence $\sigma(T) \subset \overline{\mathbb{D}}$. We can thus define $\varphi(T)$ for any rational function with poles outside $\overline{\mathbb{D}}$. Furthermore (4.2) holds as well for such functions, by approximation.

We fix two numbers $\frac{\pi}{2} < \theta < \theta' < \pi$ and we let (see Figure II.2):

$$\mathbb{D}_\theta = D\Big(-i\cot(\theta), \tfrac{1}{\sin(\theta)}\Big) \cup D\Big(i\cot(\theta), \tfrac{1}{\sin(\theta)}\Big).$$

Clearly $\mathbb{D}_\theta$ contains $\mathbb{D}$. For any $t \in (-\pi, 0) \cup (0, \pi)$, let $r(t)$ denote the radius of the largest open disc centered at e^{it} and included in $\mathbb{D}_\theta$. If t is positive and small enough, we have

$$\begin{aligned} r(t) &= \frac{1}{\sin(\theta)} - |e^{it} + i\cot(\theta)| \\ &= \frac{1}{\sin(\theta)} - \sqrt{\cos^2(t) + \big(\sin(t) + \cot(\theta)\big)^2} \\ &= \frac{1}{\sin(\theta)}\Big(1 - \sqrt{1 + 2\sin(t)\sin(\theta)\cos(\theta)}\Big) \\ &= -\cos(\theta)t + \frac{1}{2}\Big(\sin(\theta)\cos^2(\theta)\Big)t^2 + O(t^3). \end{aligned}$$

Consequently, we have $r(t) > -\cos(\theta)t$ for $t > 0$ small enough. We deduce that if $j < 0$ with $|j|$ large enough and $t \in I_j$, we have

$$D\Big(e^{it}, -\cos(\theta)\frac{\pi}{2^{|j|+1}}\Big) \subset \mathbb{D}_\theta. \tag{4.3}$$

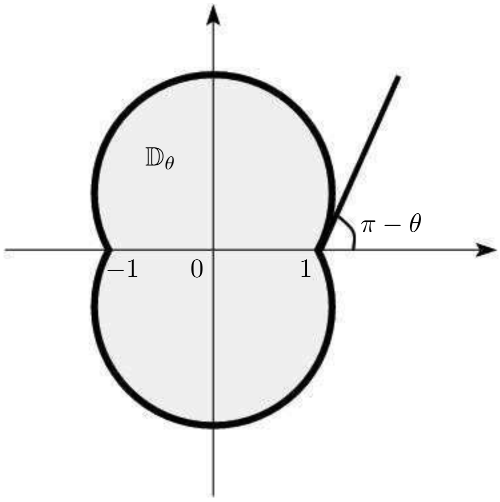

Figure II.2

The same holds for $t \in -I_j$. Moreover the intervals I_j and I_{-j} of the dyadic decomposition have length equal to $\frac{\pi}{2^{|j|+1}}$. Hence for any rational function φ with poles outside $\overline{\mathbb{D}_\theta}$ and any $j<0$ with $|j|$ large enough, we obtain that

$$\begin{aligned}
\operatorname{var}\left(\varphi_{|\mathbb{T}}, \Delta_j\right) &= \int_{-I_j \cup I_j} |\varphi'(e^{it})|\, dt \\
&\leqslant \int_{-I_j \cup I_j} \frac{\|\varphi\|_{H^\infty(\mathbb{D}_\theta)}}{-\cos(\theta)\frac{\pi}{2^{|j|+1}}}\, dt \qquad \text{by (4.3) and Cauchy's inequalities,} \\
&\leqslant \frac{\pi}{2^{|j|}} \cdot \frac{\|\varphi\|_{H^\infty(\mathbb{D}_\theta)}}{-\cos(\theta)\frac{\pi}{2^{|j|+1}}} = \frac{2\pi\|\varphi\|_{H^\infty(\mathbb{D}_\theta)}}{-\cos(\theta)}.
\end{aligned}$$

We have a similar result if $j \geqslant 0$ and large enough. Applying Theorem 4.4, we deduce a uniform estimate

$$\|\varphi(S)\|_{B(\ell^p_{\mathbb{Z}})} \lesssim \|\varphi\|_{H^\infty(\mathbb{D}_\theta)}.$$

Combining with (4.2) -as explained at the beginning of this proof- we obtain the existence of a constant $K \geqslant 0$ such that for any rational function φ with poles outside $\overline{\mathbb{D}_\theta}$,

$$\|\varphi(T)\|_{B(X)} \leqslant K\|\varphi\|_{H^\infty(\mathbb{D}_\theta)}.$$

Note that we have the following inclusion:

$$1 - \mathbb{D}_\theta \subset \Sigma_\theta.$$

Then let $\mathbb{R}_{\theta'}$ be the algebra of all rational functions with poles outside $\overline{\Sigma_{\theta'}}$ and with a nonpositive degree. We deduce from above that for any $f \in \mathbb{R}_{\theta'}$,

$$\|f(I-T)\| \leqslant K \|f(1-\cdot\,)\|_{H^\infty(\mathbb{D}_\theta)} \leqslant K\|f\|_{H^\infty(\Sigma_\theta)}.$$

According to [65, Proposition 2.10], this readily implies that $I-T$ is sectorial and admits a bounded $H^\infty(\Sigma_{\theta'})$ functional calculus. ■

Theorem 4.8 *Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. The following assertions are equivalent.*

(i) The operator T and its adjoint $T^*\colon L^{p^*}(\Omega) \to L^{p^*}(\Omega)$ both satisfy uniform estimates

$$\|x\|_{T,1} \lesssim \|x\|_{L^p} \qquad \textit{and} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}}$$

for $x \in L^p(\Omega)$ and $y \in L^{p^*}(\Omega)$.

(ii) The operator T is R-Ritt and admits a loose dilation.

(iii) The operator T is R-Ritt and p-polynomially bounded.

Proof : That (ii) implies (iii) follows from Proposition 4.6.

Assume (iii). By Proposition 4.7, $I-T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. Since T is R-Ritt, this implies (i) by Theorem 3.1.

Assume (i). It follows from [68] (see Theorem 3.1) that T is an R-Ritt operator. Thus we only need to establish the dilation property of T. Since T is R-Ritt, Theorem 3.3 ensures that the square functions $\|\ \|_{T,1}$ and $\|\ \|_{T,\frac{1}{2}}$ are equivalent on $L^p(\Omega)$. Likewise, $\|\ \|_{T^*,1}$ and $\|\ \|_{T^*,\frac{1}{2}}$ are equivalent on $L^{p^*}(\Omega)$. Consequently the assumption (i) implies the existence of a constant $C \geqslant 1$ such that

$$\|x\|_{T,\frac{1}{2}} \leqslant C\|x\|_{L^p} \qquad \text{and} \qquad \|y\|_{T^*,\frac{1}{2}} \leqslant C\|y\|_{L^{p^*}}$$

for any $x \in L^p(\Omega)$ and any $y \in L^{p^*}(\Omega)$.

We will use the direct sum decomposition (3.1), as well as the analogous decompostion of $L^{p^*}(\Omega)$ corresponding to T^*. According to the above estimates, we may define two bounded maps

$$j_1\colon \overline{\mathrm{Ran}(I-T)} \longrightarrow L^p(\Omega;\ell^2_{\mathbb{Z}}) \qquad \text{and} \qquad j_2\colon \overline{\mathrm{Ran}(I-T^*)} \longrightarrow L^{p^*}(\Omega;\ell^2_{\mathbb{Z}})$$

as follows. For any $x \in \overline{\mathrm{Ran}(I-T)}$ and any $y \in \overline{\mathrm{Ran}(I-T^*)}$, we set

$$x_k = T^{k-1}(I-T)^{\frac{1}{2}}x \qquad \text{and} \qquad y_k = (T^*)^{k-1}(I-T^*)^{\frac{1}{2}}y$$

if $k \geqslant 0$, we set $x_k = 0$ and $y_k = 0$ if $k < 0$. Then we set

$$j_1(x) = (x_k)_{k\in\mathbb{Z}} \qquad \text{and} \qquad j_2(y) = (y_k)_{k\in\mathbb{Z}}.$$

Then we let $J_1 \colon L^p(\Omega) \to L^p(\Omega) \overset{p}{\oplus} L^p(\Omega;\ell^2_{\mathbb{Z}})$ be the linear map taking any $x \in \mathrm{Ker}(I-T)$ to $(x,0)$ and any $x \in \overline{\mathrm{Ran}(I-T)}$ to $(0, j_1(x))$. We define $J_2 \colon L^{p^*}(\Omega) \to L^{p^*}(\Omega) \overset{p^*}{\oplus} L^{p^*}(\Omega;\ell^2_{\mathbb{Z}})$ in a similar way.

For any $x \in \overline{\mathrm{Ran}(I-T)}$ and $y \in \overline{\mathrm{Ran}(I-T^*)}$, we have

$$\begin{aligned}
\langle J_1 x, J_2 y\rangle \;&=\; \sum_{k=1}^{\infty} \Big\langle T^{k-1}(I-T)^{\frac{1}{2}}x, (T^*)^{k-1}(I-T^*)^{\frac{1}{2}}y\Big\rangle \\
&=\; \sum_{k=1}^{\infty} \Big\langle T^{2(k-1)}(I-T)x, y\Big\rangle \\
&=\; \sum_{k=1}^{\infty} \Big\langle T^{2(k-1)}(I-T^2)(I+T)^{-1}x, y\Big\rangle.
\end{aligned}$$

For any integer $N \geqslant 1$

$$\sum_{k=1}^{N} T^{2(k-1)}(I-T^2) \;=\; I - T^{2N}.$$

Furthermore, $(I+T)^{-1}x$ belongs to $\overline{\mathrm{Ran}(I-T)}$ and the sequence $(T^n)_{n\geqslant 0}$ strongly converges to 0 on that subspace of $L^p(\Omega)$. Hence

$$\langle J_1 x, J_2 y\rangle \;=\; \Big\langle (I+T)^{-1}x, y\Big\rangle.$$

Let $\Theta \colon L^p(\Omega) \to L^p(\Omega)$ be the linear map taking any $x \in \overline{\mathrm{Ran}(I-T)}$ to $(I+T)x$ and any $x \in \mathrm{Ker}(I-T)$ to itself. Then it follows from the above calculation that

$$\Theta J_2^* J_1 \;=\; I_{L^p(\Omega)}. \tag{4.4}$$

Let

$$Z = L^p(\Omega) \overset{p}{\oplus} L^p(\Omega;\ell^2_{\mathbb{Z}}),$$

and let $U \colon Z \to Z$ be the linear map which takes any $x \in L^p(\Omega)$ to itself and any sequence $(x_k)_{k\in\mathbb{Z}}$ in $L^p(\Omega;\ell^2_{\mathbb{Z}})$ to the shifted sequence $(x_{k+1})_{k\in\mathbb{Z}}$. Next let $P \colon Z \to Z$ be the linear map which takes any $x \in L^p(\Omega)$ to itself and any sequence $(x_k)_{k\in\mathbb{Z}}$ in $L^p(\Omega;\ell^2_{\mathbb{Z}})$ to the truncated sequence $(\ldots, 0, \ldots, 0, x_0, x_1, \ldots, x_k, \ldots)$. By construction, we have

$$PU^n J_1 = J_1 T^n, \qquad n \geqslant 0. \tag{4.5}$$

We also have $J_2^* P = J_2^*$ hence setting $J = J_1 \colon L^p(\Omega) \to Z$ and $Q = \Theta J_2^* \colon Z \to L^p(\Omega)$, we deduce from (4.4) and (4.5) that $T^n = QU^nJ$ for any $n \geqslant 0$. Furthermore, U is an isometric isomorphism on Z. Thus we have established that T satisfies the dilation property stated in Definition 4.1, except that the dilation space is Z instead of being an L^p-space.

It is easy to modify the construction to obtain a dilation through an L^p-space, as follows. First recall that using for example Gaussian variables, one can isometrically represent $\ell^2_{\mathbb{Z}}$ as a subspace of an L^p-space such that there exists a bounded projection from this space onto $\ell^2_{\mathbb{Z}}$ (see e.g. [92, Chapter 5]). The space Z can be therefore represented as well as a complemented subspace of an L^p-space. Thus we have

$$Z \oplus W \,=\, L^p(\widetilde{\Omega})$$

for an appropriate measure space $(\widetilde{\Omega}, \widetilde{\mu})$ and some Banach space W. Let $J'\colon L^p(\Omega) \to L^p(\widetilde{\Omega})$ be defined by $J'(x) = (J(x), 0)$, let $U'\colon L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ be defined by $U'(z,w) = (Uz, w)$ and let $Q'\colon L^p(\widetilde{\Omega}) \to L^p(\Omega)$ be defined by $Q'(z,w) = Q(z)$. Then U' is an isomorphism, $(U'^n)_{n\in\mathbb{Z}}$ is bounded and $Q'U'^n J' = T^n$ for any $n \geqslant 0$. ∎

5 Comparing p-boundedness properties

In this section we will consider an L^p-analog of complete polynomial boundedness going back to [93] (see also [97, Chapter 8]) and give complements to the results obtained in the previous section. In particular we will show the existence of p-polynomially bounded operators $L^p \to L^p$ without any loose dilation.

In the sequel we assume that $1 \leqslant p < \infty$. Let $n \geqslant 1$ be an integer. For any vector space V, we let $M_n(V)$ denote the space of $n \times n$ matrices with entries in V. When $V = B(X)$ for some Banach space X, we equip this space with a specific norm, as follows. For any $[T_{ij}]_{1\leqslant i,j\leqslant n}$ in $M_n(B(X))$, we set

$$\big\|[T_{ij}]\big\|_{p,M_n(B(X))} \,=\, \sup\left\{\bigg(\sum_{i=1}^n \Big\|\sum_{j=1}^n T_{ij}(x_j)\Big\|_X^p\bigg)^{\frac{1}{p}} \,:\, x_1,\ldots,x_n \in X,\ \sum_{j=1}^n \|x_j\|_X^p \leqslant 1\right\}. \tag{5.1}$$

In other words, we regard $[T_{ij}]$ as an operator $\ell^p_n(X) \to \ell^p_n(X)$ in a natural way and the norm of the matrix is defined as the corresponding operator norm.

Let X, Y be two Banach spaces, let $V \subset B(X)$ be a subspace and let $u\colon V \to B(Y)$ be a linear mapping. We say that u is p-completely bounded if there exists a constant $C \geqslant 0$ such that

$$\big\|[u(T_{ij})]\big\|_{p,M_n(B(Y))} \leqslant C\big\|[T_{ij}]\big\|_{p,M_n(B(X))}$$

for any $n \geqslant 1$ and any matrix $[T_{ij}]$ in $M_n(V)$. In this case, we let $\|u\|_{pcb}$ denote the smallest possible C.

Let us regard the vector space $\mathcal{P}$ of all complex polynomials as a subspace of $B(\ell^p_{\mathbb{Z}})$, by identifying any $\varphi \in \mathcal{P}$ with the operator $\varphi(S)$. Accordingly for any $[\varphi_{ij}]$ in $M_n(\mathcal{P})$, we set

$$\big\|[\varphi_{ij}]\big\|_p \,=\, \big\|[\varphi_{ij}(S)]\big\|_{p,M_n(B(\ell^p_{\mathbb{Z}}))}.$$

This extends (4.1) to matrices. We say that a bounded operator $T\colon Y \to Y$ is p-completely polynomially bounded if the natural mapping $u\colon \mathcal{P} \to B(Y)$ given by $u(\varphi) = \varphi(T)$ is p-completely bounded.

This is equivalent to the existence of a constant $C \geqslant 1$ such that

$$\big\|[\varphi_{ij}(T)]\big\|_{p,M_n(B(Y))} \leqslant C\big\|[\varphi_{ij}]\big\|_p$$

for any matrix $[\varphi_{ij}]$ of complex polynomials.

When $p = 2$ and Y is a Hilbert space, the notions of 2-polynomial boundedness and 2-complete polynomial boundedness correspond to the usual notions of polynomial boundedness and complete polynomial boundedness from [86, 87]. See [86] for the rich connections with operator space theory. The existence of a polynomially bounded operator on Hilbert space which is not completely polynomially bounded is a major result due to Pisier. Indeed this is the heart of his negative solution to the Halmos problem [97, 98]. We will show that Pisier's construction can be transferred to our L^p-setting.

We start with an elementary result which is obvious when $p = 2$ but requires attention when $p \neq 2$.

Lemma 5.1 *Let $N \geqslant 1$ be an integer, let H be a Hilbert space and let $\pi \colon B(\ell^2_N) \to B(H)$ be a unital $*$-representation. Then for any $n \geqslant 1$ and any matrix $[T_{ij}]$ in $M_n(B(\ell^2_N))$, we have*

$$\big\|[T_{ij}]\big\|_{p,M_n(B(\ell^2_N))} \leqslant \big\|[\pi(T_{ij})]\big\|_{p,M_n(B(H))}.$$

Proof : As is well-known, there is a Hilbert space K such that

$$H \simeq \ell^2_N(K), \qquad B(H) \simeq B(\ell^2_N) \otimes B(K),$$

and $\pi(T) = T \otimes I_K$ for any $T \in B(\ell^2_N)$ (see e.g. [27, Corollary III.1.7]). Consider $[T_{ij}]$ in $M_n\big(B(\ell^2_N)\big)$ and $x_1, \ldots, x_n$ in ℓ^2_N. Fix some $e \in K$ with $\|e\| = 1$. Then

$$\begin{aligned} \sum_i \Big\| \sum_j T_{ij}(x_j) \Big\|^p_{\ell^2_N} &= \sum_i \Big\| \sum_j T_{ij}(x_j) \otimes e \Big\|^p_{\ell^2_N(K)} \\ &= \sum_i \Big\| \sum_j [\pi(T_{ij})](x_j \otimes e) \Big\|^p_{\ell^2_N(K)} \\ &\leqslant \big\|[\pi(T_{ij})]\big\|^p_{p,M_n(B(H))} \sum_j \|x_j \otimes e\|^p_{\ell^2_N(K)} \\ &\leqslant \big\|[\pi(T_{ij})]\big\|^p_{p,M_n(B(H))} \sum_j \|x_j\|^p_{\ell^2_N}\,, \end{aligned}$$

and the result follows at once. ∎

Proposition 5.2 *Suppose that $1 < p < \infty$. There exists a p-polynomially bounded operator T on $L^p([0,1])$ which is not p-completely polynomially bounded.*

Proof : We need some background on Pisier's counterexample. We refer to [97, Chapter 9] and [86, Chapter 10] for a detailed exposition of this example and also to the necessary background on Hankel operators on $B(\ell^2(H))$ and their $B(H)$-valued symbols.

We start with a concrete description of a sequence of operators satisfying the so-called canonical anticommutation relations. Let I_2 denote the identity matrix on M_2. For any $k \geqslant 1$, consider the unital embedding $M_{2^k} \hookrightarrow M_{2^{k+1}} \simeq M_{2^k} \otimes M_2$ given by $A \mapsto A \otimes I_2$. The closure of the union of the resulting increasing sequence $(M_{2^k})_{k\geqslant 1}$ is a C^*-algebra. Representing it as an algebra of operators, we obtain a Hilbert space H and an embedding

$$\bigcup_{k\geqslant 1} \uparrow M_{2^k} \subset B(H) \tag{5.2}$$

whose restriction to each M_{2^k} is a unital $*$-representation.

Consider the 2×2 matrices

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For any $k \geqslant 1$, we set

$$C_k = E^{\otimes(k-1)} \otimes D \ \in M_{2^k},$$

where $E^{\otimes(k-1)}$ denotes the tensor product of E with itself $(k-1)$ times. Then following (5.2) we let $\widetilde{C_k}$ denote this operator regarded as an element of $B(H)$. The distinction between C_k and $\widetilde{C_k}$ may look superfluous. The reason why we need this is that the inclusion providing the identification between C_k and $\widetilde{C_k}$ is a $*$-representation and a priori, $*$-representations are not p-complete isometries (i.e. they do not preserve p-matrix norms). However using Lemma 5.1, we see that for any $m \geqslant 1$, for any $n \geqslant 1$ and for any $a_1, \ldots, a_m \in M_n$,

$$\Big\| \sum_{k=1}^{m} a_k \otimes C_k \otimes I_2^{\otimes(m-k)} \Big\|_{p, M_n(B(\ell^2_{2^m}))} \leqslant \Big\| \sum_{k=1}^{m} a_k \otimes \widetilde{C_k} \Big\|_{p, M_n(B(H))}. \tag{5.3}$$

The above sequence of matrices has the following remarkable property (see [97, page 70]): for any complex numbers $\alpha_1, \ldots, \alpha_m$,

$$\Big\| \sum_{k=1}^{m} \alpha_k\, C_k \otimes I_2^{\otimes(m-k)} \Big\|_{B(\ell^2_{2^m})} = \Big(\sum_{k=1}^{m} |\alpha_k|^2 \Big)^{\frac{1}{2}}. \tag{5.4}$$

Let $\mathcal{H} = \ell^2(H) \overset{2}{\oplus} \ell^2(H)$, let $\sigma \colon \ell^2(H) \to \ell^2(H)$ denote the shift operator, let $\Gamma \colon \ell^2(H) \to \ell^2(H)$ be the Hankel operator associated to the $B(H)$-valued function F given by

$$F(t) = \sum_{k=1}^{\infty} \frac{\widetilde{C_k}}{2^k}\, e^{-i(2^k-1)t},$$

and let $T \in B(\mathcal{H})$ be the operator given by

$$T = \begin{bmatrix} \sigma^* & \Gamma \\ 0 & \sigma \end{bmatrix}.$$

Pisier proved that this operator is polynomially bounded without being completely polynomially bounded. Since $\|\cdot\|_2 \leqslant \|\cdot\|_p$ on $\mathcal{P}$, the linear mapping

$$u\colon (\mathcal{P}, \|\ \|_p) \longrightarrow B(\mathcal{H}), \qquad u(\varphi) = \varphi(T),$$

is therefore bounded. Our aim is now to show that u is not p-completely bounded.

We consider the auxiliary mapping $w\colon \mathcal{P} \to B(H)$ defined by letting

$$w\left(\sum_{k\geqslant 0} d_k z^k\right) = \sum_{k\geqslant 0} d_{2^k}\widetilde{C_k}$$

for any finite sequence $(d_k)_{k\geqslant 0}$ of complex numbers.

Let $j\colon H \to \ell^2(H)$ be the isometric embedding given by $j(x) = (x, 0, \ldots, 0, \ldots)$. Then we define the map $v\colon B(\ell^2(H)) \to B(H)$ by letting $v(R) = j^*Rj$ for any $R \in B(H)$. It is easy to check that v is p-completely bounded, with $\|v\|_{pcb} = 1$. On the other hand, for any $\varphi \in \mathcal{P}$, we have

$$\varphi(T) = \begin{bmatrix} \varphi(\sigma^*) & \Gamma\varphi'(\sigma) \\ 0 & \varphi(\sigma) \end{bmatrix},$$

see [97, (9.7)]. Let $\widetilde{u}\colon \mathcal{P} \to B(\ell^2(H))$ be defined by $\widetilde{u}(\varphi) = \Gamma\varphi'(\sigma)$. Then the argument in the proof of [97, Theorem. 9.7] shows that $w = v\widetilde{u}$. Thus if u were p-completely bounded, then w would be p-completely bounded as well. Let us show that this does not hold true.

Note that for any Banach space X, for any integer $N \geqslant 1$, for any $T \in B(X)$ and for any $A \in B(\ell^1_N)$, we have

$$\|A \otimes T\|_{\ell^1_N(X)\to\ell^1_N(X)} = \|A\|_{B(\ell^1_N)}\|T\|_{B(X)}.$$

This can be be seen as a consequence of the fact that $\ell^1_N(X)$ is the projective tensor product of ℓ^1_N and X, see [33, Chapter VIII], however an elementary proof is also possible (we leave this to the reader).

Let $m \geqslant 1$. Clearly $\|E\|_{B(\ell^1_2)} = \|D\|_{B(\ell^1_2)} = 1$. Hence applying the above property we have $\|C_k\|_{B(\ell^1_{2^k})} = \|E\|^{k-1}_{B(\ell^1_2)}\|D\|_{B(\ell^1_2)} = 1$ and hence

$$\left\|C_k \otimes I_2^{\otimes(m-k)} \otimes S^{2^k}\right\|_{B(\ell^1_{2^m}(\ell^1_{\mathbb{Z}}))} = 1, \qquad k = 1, \ldots, n. \tag{5.5}$$

Let $\varphi_m \in M_{2^m} \otimes \mathcal{P}$ be given by

$$\varphi_m(z) = \sum_{k=1}^{m} C_k \otimes I_2^{\otimes(m-k)}\, z^{2^k}.$$

By (5.4), we have

$$\|\varphi_m\|_2 = \sup_{|z|=1} \|\varphi_m(z)\|_{B(\ell^2_{2^m})} = \sup_{|z|=1} \Big\| \sum_{k=1}^{m} z^{2^k}\, C_k \otimes I_2^{\otimes(m-k)} \Big\|_{B(\ell^2_{2^m})} = \sqrt{m}.$$

On the other hand, applying (5.5) we have

$$\|\varphi_m\|_1 \,=\, \Big\| \sum_{k=1}^{m} C_k \otimes I_2^{\otimes(m-k)} \otimes S^{2^k} \Big\|_{B(\ell^1_{2^m}(\ell^1_{\mathbb{Z}}))} \,\leqslant\, \sum_{k=1}^{m} \Big\| C_k \otimes I_2^{\otimes(m-k)} \otimes S^{2^k} \Big\|_{B(\ell^1_{2^m}(\ell^1_{\mathbb{Z}}))} \,=\, m.$$

By interpolation, we deduce that

$$\|\varphi_m\|_p \leqslant m^{\frac{1}{p}}. \tag{5.6}$$

Next we have

$$\Big(I_{B(\ell^p_{2^m})} \otimes w\Big)(\varphi_m) \,=\, \sum_{k=1}^{m} C_k \otimes I_2^{\otimes(m-k)} \otimes \widetilde{C_k}\,. \tag{5.7}$$

Let us estimate the norm of this tensor product in $B(\ell^p_{2^m}(H))$. Let (e_1, e_2) denote the canonical basis of $\mathbb{C}^2$. For any $k = 1, \ldots, m$ and any $i_1, \ldots, i_m, j_1, \ldots, j_m$ in $\{1, 2\}$,

$$\begin{aligned}
&\Big\langle \Big(C_k \otimes I_2^{\otimes(m-k)}\Big)(e_{j_1} \otimes \cdots \otimes e_{j_m}), e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\rangle \\
&\quad = \Big\langle E e_{j_1} \otimes \cdots \otimes E e_{j_{k-1}} \otimes D e_{j_k} \otimes e_{j_{k+1}} \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\rangle \\
&\quad = \Big\langle (-1)^{\delta_{j_1,2}} e_{j_1} \otimes \cdots \otimes (-1)^{\delta_{j_{k-1},2}} e_{j_{k-1}} \otimes \delta_{j_k,2} e_1 \otimes e_{j_{k+1}} \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\rangle \\
&\quad = (-1)^{\delta_{j_1,2}} \cdots (-1)^{\delta_{j_{k-1},2}} \delta_{j_k,2} \langle e_{j_1}, e_{i_1} \rangle \cdots \langle e_{j_{k-1}}, e_{i_{k-1}} \rangle \langle e_1, e_{i_k} \rangle \langle e_{j_{k+1}}, e_{i_{k+1}} \rangle \cdots \langle e_{j_m}, e_{i_m} \rangle \\
&\quad = (-1)^{\delta_{j_1,2}} \cdots (-1)^{\delta_{j_{k-1},2}} \delta_{j_k,2} \delta_{1,i_k} \delta_{j_1,i_1} \cdots \delta_{j_{k-1},i_{k-1}} \delta_{j_{k+1},i_{k+1}} \cdots \delta_{j_m,i_m}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Big\langle \Big(\sum_{k=1}^{m} \Big(C_k \otimes I_2^{\otimes(m-k)}\Big) \otimes \Big(C_k \otimes I_2^{\otimes(m-k)}\Big) \Big) \Big(\sum_{j_1,\ldots,j_m=1}^{2} e_{j_1} \otimes \cdots \otimes e_{j_m} \otimes e_{j_1} \otimes \cdots \otimes e_{j_m} \Big), \\
\sum_{i_1,\ldots,i_m=1}^{2} e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\rangle
\end{aligned}$$

is equal to

$$\sum_{k=1}^{m}\sum_{\substack{j_1,\dots,j_m,\\ i_1,\dots,i_m=1}}^{2}\left\langle\Big(C_k\otimes I_2^{\otimes(m-k)}\Big)(e_{j_1}\otimes\cdots\otimes e_{j_m}),e_{i_1}\otimes\cdots\otimes e_{i_m}\right\rangle^2$$
$$=\sum_{k=1}^{m}\sum_{\substack{j_1,\dots,j_m,\\ i_1,\dots,i_m=1}}^{2}\Big(\delta_{j_k,2}\delta_{1,i_k}\delta_{j_1,i_1}\cdots\delta_{j_{k-1},i_{k-1}}\delta_{j_{k+1},i_{k+1}}\cdots\delta_{j_m,i_m}\Big)^2$$
$$=\sum_{k=1}^{m}2^{m-1} \,=\, m2^{m-1}.$$

Since the norm of

$$\sum_{i_1,\dots,i_m=1}^{2} e_{i_1}\otimes\cdots\otimes e_{i_m}\otimes e_{i_1}\otimes\cdots\otimes e_{i_m}$$

in $\ell^p_{2^m}(\ell^2_{2^m})$ (resp. in $\ell^{p^*}_{2^m}(\ell^2_{2^m})$) is equal to

$$\left(\sum_{i_1,\dots,i_m=1}^{2}\|e_{i_1}\otimes\cdots\otimes e_{i_m}\|^p_{\ell^2_{2^m}}\right)^{\frac{1}{p}} \,=\, 2^{\frac{m}{p}}$$

(resp. $2^{\frac{m}{p^*}}$), we deduce that

$$\left\|\sum_{k=1}^{m}\Big(C_k\otimes I_2^{\otimes(m-k)}\Big)\otimes\Big(C_k\otimes I_2^{\otimes(m-k)}\Big)\right\|_{B(\ell^p_{2^m}(\ell^2_{2^m}))}\geqslant\frac{m}{2}.$$

Combining with (5.3) and (5.7) we obtain that

$$\Big\|(I_{B(\ell^p_{2^m})}\otimes w)(\varphi_m)\Big\|\geqslant\frac{m}{2}\,.$$

Together with (5.6), this implies that w is not p-completely bounded. Thus T is not p-completely polynomially bounded.

So we are done except that T acts on the Hilbert space $\mathcal{H}$ and not on $L^p([0,1])$. However arguing as in the last part of the proof of Theorem 4.8, it is easy to pass from $\mathcal{H}$ to the space $L^p([0,1])$. ∎

The proof of Proposition 4.6 actually yields the following stronger result: if an operator $T\colon L^p(\Omega)\to L^p(\Omega)$ admits a loose dilation, then it is p-completely polynomially bounded (details are left to the reader). Hence the above proposition yields the following.

Corollary 5.3 *There exists a p-polynomially bounded operator $T\colon L^p([0,1])\to L^p([0,1])$ which does not admit any loose dilation.*

Note also that according to Theorem 4.8 and the above observation, no R-Ritt operator can satisfy Proposition 5.2. Namely, if $T\colon L^p(\Omega)\to L^p(\Omega)$ is an R-Ritt operator and is p-polynomially bounded, then it is p-completely polynomially bounded.

Remark 4.3 and the above investigations lead to the following open problem (for $p \neq 2$): *does any p-completely polynomially bounded operator $T\colon L^p(\Omega) \to L^p(\Omega)$ admit a loose dilation?*

In the last part of this section we are going to consider another type of counterexamples. Clearly any p-polynomially bounded $T\colon L^p(\Omega) \to L^p(\Omega)$ is automatically power bounded, that is,

$$\sup_{n\geqslant 0} \|T^n\| < \infty\,.$$

The existence of a power bounded operator on Hilbert space which is not polynomially bounded is an old result of Foguel [42]. Our aim is to prove an L^p-analog of that result. We will actually show a stronger form: there exists a Ritt operator which is not p-polynomialy bounded. To achieve this, we will adapt the approach used in [55] to go beyond Foguel's Theorem.

We need some background on Schauder bases and their multipliers that we briefly recall. We let v_1 denote the set of all sequences $(c_n)_{n\geqslant 0}$ of complex numbers whose variation $\sum_{n=1}^{\infty} |c_n - c_{n-1}|$ is finite. Any such sequence is bounded and v_1 is a Banach space for the norm

$$\big\|(c_n)_{n\geqslant 0}\big\|_{v_1} = |c_0| + \sum_{n=1}^{\infty} |c_n - c_{n-1}|\,.$$

Let $(e_n)_{n\geqslant 0}$ be a Schauder basis on some Banach space X. For any $n \geqslant 0$, let $Q_n\colon X \to X$ be the projection defined by

$$Q_n\Bigg(\sum_{k=0}^{\infty} a_k e_k\Bigg) = \sum_{k=n}^{\infty} a_k e_k \tag{5.8}$$

for any converging sequence $\sum_k a_k e_k$. The sequence $(Q_n)_{n\geqslant 0}$ is bounded and by a standard Abel summation argument, we have the following.

Lemma 5.4 *For any $c = (c_n)_{n\geqslant 0}$ in v_1, there exists a (necessarily unique) bounded operator $T_c\colon X \to X$ such that*

$$T_c\Bigg(\sum_{n=0}^{\infty} a_n e_n\Bigg) = \sum_{n=0}^{\infty} c_n a_n e_n$$

for any converging sequence $\sum_n a_n e_n$. Furthermore,

$$\|T_c\| \leqslant \Big(\sup_{n\geqslant 0} \|Q_n\|\Big) \big\|(c_n)_{n\geqslant 1}\big\|_{v_1}.$$

The above operator T_c is called the multiplier associated to the sequence c.

Proposition 5.5 *Let $1 < p < \infty$.*

(1) There exists a Ritt (hence a power bounded) operator on ℓ^2 which is not p-polynomially bounded.

(2) There exists an R-Ritt operator on $L^p([0,1])$ which is not p-polynomially bounded.

Proof : (1): We let $(e_n)_{n\geqslant 0}$ be a Schauder basis of $H=\ell^2$. It is clear that the sequence $\bigl(1-\frac{1}{2^n}\bigr)_{n\geqslant 0}$ has a finite variation. According to the above discussion, we let $T\colon H\to H$ denote the multiplier associated to that sequence.

For any $\theta\in(-\pi,0)\cup(0,\pi]$, set

$$c(\theta)_n = \frac{1}{e^{i\theta}-(1-\frac{1}{2^n})}\,, \qquad n\geqslant 0.$$

We have

$$\begin{aligned}\sum_{n=1}^{+\infty}|c(\theta)_n - c(\theta)_{n-1}| &= \sum_{n=1}^{+\infty}\left|\int_{1-\frac{1}{2^{n-1}}}^{1-\frac{1}{2^n}}\frac{dt}{\bigl(e^{i\theta}-t\bigr)^2}\right|\\ &\leqslant \sum_{n=1}^{+\infty}\int_{1-\frac{1}{2^{n-1}}}^{1-\frac{1}{2^n}}\frac{dt}{|e^{i\theta}-t|^2}\\ &\leqslant \int_0^1\frac{dt}{|e^{i\theta}-t|^2}\,.\end{aligned}$$

Let $I(\theta)$ denote the latter integral. It is finite hence $c(\theta)=(c(\theta)_n)_{n\geqslant 0}$ belongs to v_1. It is easy to deduce that $e^{i\theta}-T$ is invertible, the operator $R(e^{i\theta},T)$ being the multiplier associated to the sequence $c(\theta)$.

For $\theta\neq\pi$, elementary computations yield

$$\begin{aligned}I(\theta) &= \int_0^1\frac{dt}{(t-\cos(\theta))^2+\sin^2(\theta)}\\ &= \frac{1}{\sin(\theta)}\int_{-\frac{\cos(\theta)}{\sin(\theta)}}^{\frac{1-\cos(\theta)}{\sin(\theta)}}\frac{du}{1+u^2}\\ &= \frac{\pi-\theta}{2\sin(\theta)}\,.\end{aligned}$$

Moreover $|e^{i\theta}-1| = 2\sin\bigl(\frac{\theta}{2}\bigr)$, hence

$$|e^{i\theta}-1|I(\theta) \,=\, (\pi-\theta)\frac{\sin\bigl(\frac{\theta}{2}\bigr)}{\sin(\theta)} \,=\, \frac{\pi-\theta}{2\cos\bigl(\frac{\theta}{2}\bigr)}\,.$$

This is bounded for θ varying in $(-\pi,0)\cup(0,\pi)$. According to Lemma 5.4, this shows that

$$\sigma(T)\subset\mathbb{D}\cup\{1\}\qquad\text{and}\qquad\Bigl\{(\lambda-1)R(\lambda,T)\,:\,\lambda\in\mathbb{T}\setminus\{1\}\Bigr\}\quad\text{is bounded.}$$

Applying the maximum principle to the function $z\mapsto(1-z)(I_H-zT)^{-1}$, we deduce that the set $\{(\lambda-1)R(\lambda,T)\,:\,|\lambda|>1\}$ is bounded as well, and hence T is a Ritt operator.

Let us now assume that the basis $(e_n)_{n\geqslant 0}$ is not an unconditional one. The operator $I-T$ is the multiplier associated to the sequence $\bigl(\frac{1}{2^n}\bigr)_{n\geqslant 0}$ and as is well-known, the lack of unconditionality implies that for any $\theta \in (0,\pi)$, this operator does not have a bounded $H^\infty(\Sigma_\theta)$ functional calculus (see e.g. [65, Theorem 4.1] and its proof). According to Proposition 4.7, this implies that T is not p-polynomially bounded.

(2): Since all bounded subsets of $B(\ell^2)$ are R-bounded, the operator considered in part (1) is automatically an R-Ritt operator. Then arguing again as in the proof of Theorem 4.8, it is easy to pass from an ℓ^2-operator to an $L^p([0,1])$-operator which is not p-polynomially bounded although being an R-Ritt operator. ∎

6 Extensions to general Banach spaces

Up to now we have mostly dealt with operators acting on (commutative) L^p-spaces. In this last section, we shall consider more general Banach spaces, in particular noncommutative L^p-spaces. We aim at extending our main results from Sections 3 and 4 to this broader context.

We will use classical notions from Banach space theory such as cotype, K-convexity and the UMD property. We refer the reader to [21, 32, 92] for background.

In accordance with (2.1), we are going to extend the definitions (1.1) and (1.3) to arbitrary Banach spaces using Rademacher averages. Recall Section 2 for notation. The use of such averages as a substitute of square functions on abstract Banach spaces is a classical and fruitful principle. See e.g. [52, 54, 68].

Let X be a Banach space, let $T\colon X \to X$ be any bounded operator and let $x \in X$. Consider the element $x_k = k^{\frac{1}{2}}(T^k(x) - T^{k-1}(x))$ for any $k \geqslant 1$. If the series $\sum_k \varepsilon_k \otimes x_k$ converges in $L^2(\Omega_0; X)$ then we set

$$\|x\|_{T,1} = \Bigl\| \sum_{k=1}^{+\infty} k^{\frac{1}{2}}\, \varepsilon_k \otimes \bigl(T^k(x) - T^{k-1}(x)\bigr) \Bigr\|_{\mathrm{Rad}(X)}.$$

We set $\|x\|_{T,1} = \infty$ otherwise. Likewise, if T is a Ritt operator and $\alpha > 0$ is a positive real number, then we set

$$\|x\|_{T,\alpha} = \Bigl\| \sum_{k=1}^{+\infty} k^{\alpha-\frac{1}{2}}\, \varepsilon_k \otimes T^{k-1}(I-T)^\alpha x \Bigr\|_{\mathrm{Rad}(X)}$$

if the corresponding series converges in $L^2(\Omega_0; X)$, and $\|x\|_{T,\alpha} = \infty$ otherwise. The following extends Theorem 3.3.

Theorem 6.1 *Assume that X is reflexive and has a finite cotype. Let $T\colon X \to X$ be an R-Ritt operator. Then for any $\alpha > 0$ and $\beta > 0$, we have an equivalence*

$$\|x\|_{T,\alpha} \approx \|x\|_{T,\beta}, \qquad x \in X.$$

Proof : We noticed in Section 2 that if X has a finite cotype, then Rademacher averages and Gaussian averages are equivalent on X.

Furthermore, the reflexivity of X ensures that it satisfies the Mean Ergodic Theorem. We thus have
$$X = \operatorname{Ker}(I-T) \oplus \overline{\operatorname{Ran}(I-T)}.$$

Lastly, since X has a finite cotype, it cannot contain c_0 (as an isomorphic subspace). Hence by [61], a series $\sum_k \varepsilon_k \otimes x_k$ converges in $L^2(\Omega_0; X)$ if (and only if) its partial sums are uniformly bounded, that is, there is a constant $K \geqslant 0$ such that
$$\Big\| \sum_{k=1}^{N} \varepsilon_k \otimes x_k \Big\|_{\operatorname{Rad}(X)} \leqslant K, \qquad N \geqslant 1.$$

With these three properties in hand, it is easy to see that our proof of Theorem 3.3 extends verbatim to the general case. ∎

In the rest of this section we are going to focus on noncommutative L^p-spaces. We let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace and for any $1 \leqslant p < \infty$, we let $L^p(M)$ denote the associated (noncommutative) L^p-space. We refer to [103] for background and information on these spaces. Any element of $L^p(M)$ is a (possibly unbounded) operator and for any such x, we set
$$|x| = (x^*x)^{\frac{1}{2}}.$$
We recall the noncommutative analog of (2.1) from [71] (see also [72]). For finite families $(x_k)_k$ of $L^p(M)$, we have the following equivalences. If $2 \leqslant p < \infty$, then
$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\operatorname{Rad}(L^p(M))} \approx \max\Bigg\{ \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)}, \Big\| \Big(\sum_k |x_k^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \Bigg\}. \tag{6.1}$$
If $1 < p \leqslant 2$, then
$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\operatorname{Rad}(L^p(M))} \approx \inf\Bigg\{ \Big\| \Big(\sum_k |u_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} + \Big\| \Big(\sum_k |v_k^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \Bigg\}, \tag{6.2}$$
where the infimum runs over all possible decompositions $x_k = u_k + v_k$ in $L^p(M)$.

Let $T\colon L^p(M) \to L^p(M)$ be a bounded operator. We say that T admits a noncommutative loose dilation if there exist a von Neumann algebra $\widetilde{M}$, an isomorphism $U\colon L^p(\widetilde{M}) \to L^p(\widetilde{M})$ such that the set $\{U^n : n \in \mathbb{Z}\}$ is bounded and two bounded maps $L^p(M) \xrightarrow{J} L^p(\widetilde{M})$ and $L^p(\widetilde{M}) \xrightarrow{Q} L^p(M)$ such that $T^n = QU^nJ$ for any integer $n \geqslant 0$. We say that T admits a noncommutative strict dilation if this holds true for an isometric isomorphism U and two contractions J and Q. As opposed to the commutative case (see Remark 4.2), there is no characterization of contractions $T\colon L^p(M) \to L^p(M)$ which admit a noncommutative strict dilation. The gap with the commutative situation is illustrated by the following result [51, Theorem 5.1]: for any $p \neq 2$, there exist a completely positive contraction on some finite dimensional noncommutative L^p-space with no noncommutative strict dilation.

We now turn to loose dilations. In the commutative setting, the following proposition is a combination of Propositions 4.6 and 4.7.

Proposition 6.2 *Let $T\colon L^p(M) \to L^p(M)$, with $1 < p < \infty$. If T admits a noncommutative loose dilation, then $I - T$ is sectorial and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta \in (\frac{\pi}{2}, \pi)$.*

Proof : Let us explain how to adapt the 'commutative' proof to the present setting.

First we extend the definition (4.1) as follows. For any Banach space X, let $S_X\colon \ell^p_{\mathbb{Z}}(X) \to \ell^p_{\mathbb{Z}}(X)$ denote the shift operator. Then for any $\varphi \in \mathcal{P}$, we set

$$\|\varphi\|_{p,X} = \|\varphi(S_X)\|_{B(\ell^p_{\mathbb{Z}}(X))}.$$

It follows from [15, Theorem 4.3] that if X is UMD, then Theorem 4.4 holds as well for scalar valued Fourier multipliers on $\ell^p_{\mathbb{Z}}(X)$. In this case, the argument in the proof of Proposition 4.7 leads to the following: for any $\theta \in (\frac{\pi}{2}, \pi)$, there is an estimate

$$\|\varphi\|_{p,X} \lesssim \|\varphi\|_{H^\infty(\mathbb{D}_\theta)} \tag{6.3}$$

for rational functions φ with poles outside $\overline{\mathbb{D}_\theta}$.

Second we note that if $U\colon L^p(\widetilde{M}) \to L^p(\widetilde{M})$ is an isomorphism such that $K = \sup\{\|U^n\| \,:\, n \in \mathbb{Z}\} < \infty$, then the vectorial version of the transference principle (see [9, Theorem 2.8]) ensures that for any φ as above, we have

$$\|\varphi(U)\| \leqslant K^2 \|\varphi\|_{p,L^p(\widetilde{M})}.$$

Assume now that $T\colon L^p(M) \to L^p(M)$ admits a noncommutative loose dilation. Noncommutative L^p-spaces are UMD hence property (6.3) applies to them. Hence arguing as in the proof of Proposition 4.6, we find an estimate

$$\|\varphi(T)\| \lesssim \|\varphi\|_{H^\infty(\mathbb{D}_\theta)}$$

for rational functions φ with poles outside $\overline{\mathbb{D}_\theta}$. Finally the argument at the end of the proof of Proposition 4.7 yields that $I - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. We skip the details. ∎

We are now ready to give a noncommutative analog of Theorem 4.8.

Theorem 6.3 *Let $T\colon L^p(M) \to L^p(M)$ be an R-Ritt operator, with $1 < p < \infty$.*

(1) The following assertions are equivalent.

(i) The operator T admits a noncommutative loose dilation.

(ii) The operator T and its adjoint $T^*\colon L^{p^*}(M) \to L^{p^*}(M)$ both satisfy uniform estimates

$$\|x\|_{T,1} \lesssim \|x\|_{L^p(M)} \qquad \textit{and} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}(M)}$$

for $x \in L^p(M)$ and $y \in L^{p^*}(M)$.

(2) Assume that $p \geqslant 2$. Then the above conditions are equivalent to the existence of a constant $C \geqslant 1$ for which the following two properties hold.

(iii) For any $x \in L^p(M)$,

$$\Big\| \Big(\sum_{k=1}^{+\infty} k \big| T^k(x) - T^{k-1}(x) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \leqslant C \|x\|_{L^p(M)}$$

and

$$\Big\| \Big(\sum_{k=1}^{+\infty} k \big| (T^k(x) - T^{k-1}(x))^* \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \leqslant C \|x\|_{L^p(M)}.$$

(iii) For any $y \in L^{p^*}(M)$, there exist two sequences $(u_k)_{k\geqslant 1}$ and $(v_k)_{k \geqslant 1}$ of $L^{p^*}(M)$ such that*

$$\Big\| \Big(\sum_{k=1}^{+\infty} |u_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p^*}(M)} + \Big\| \Big(\sum_{k=1}^{+\infty} |v_k^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p^*}(M)} \leqslant C \|y\|_{L^{p^*}(M)}\,,$$

and

$$u_k + v_k \ = \ k^{\frac{1}{2}} \big(T^{*k}(y) - T^{*(k-1)}(y) \big) \qquad \textit{for any } k \geqslant 1.$$

Proof : Theorem 3.1 holds as well on noncommutative L^p-spaces for R-Ritt operators, by [68]. Combining that result with Proposition 6.2, we obtain that (i) implies (ii).

Assume (ii) and suppose for simplicity that $I - T$ is 1-1 (the changes to treat the general case are minor ones). By Theorem 6.1, we have uniform estimates

$$\|x\|_{T, \frac{1}{2}} \lesssim \|x\|_{L^p(M)} \qquad \text{and} \qquad \|y\|_{T^*, \frac{1}{2}} \lesssim \|y\|_{L^{p^*}(M)}$$

for $x \in L^p(M)$ and $y \in L^{p^*}(M)$. As in the proof of Theorem 4.8, we may therefore define $J_1 \colon L^p(M) \to \mathrm{Rad}(L^p(M))$ and $J_2 \colon L^{p^*}(M) \to \mathrm{Rad}(L^{p^*}(M))$ by setting

$$J_1(x) = \sum_{k=1}^{+\infty} \varepsilon_k \otimes T^{k-1}(I-T)^{\frac{1}{2}} x \qquad \text{and} \qquad J_2(y) = \sum_{k=1}^{+\infty} \varepsilon_k \otimes T^{*(k-1)}(I-T^*)^{\frac{1}{2}} y$$

for any $x \in L^p(M)$ and any $y \in L^{p^*}(M)$. Since $L^p(M)$ is K-convex, we have a natural isomorphism

$$\Big(\mathrm{Rad}(L^p(M)) \Big)^* \approx \mathrm{Rad}(L^{p^*}(M)). \tag{6.4}$$

Hence one can consider the composition $J_2^* J_1$, it is equal to $(I+T)^{-1}$ and one obtains (i) by simply adapting the proof of Theorem 4.8.

Finally the equivalence between (ii) and (iii)+(iii)* follows from (6.1) and (6.2). ∎

Note that switching (iii) and (iii)*, we find a version of (2) for the case $p \leqslant 2$.

Remark 6.4

(1) Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be an R-Ritt operator on some commutative L^p-space. Combining Theorems 6.3 and 4.8, we find that T admits a noncommutative loose dilation (if and) only if it admits a commutative one.

(2) Proposition 3.4 holds true on noncommutative L^p-spaces. The proof is similar, using (6.4) instead of the duality $L^p(\Omega;\ell^2)^* = L^{p^*}(\Omega;\ell^2)$.

Remark 6.5 *Let $T\colon B(\ell^2) \to B(\ell^2)$ be a contractive Schur multiplier, given by*

$$[c_{ij}]_{i,j\geqslant 1} \xmapsto{T} [t_{ij}c_{ij}]_{i,j\geqslant 1}$$

for some bi-infinite matrix $[t_{ij}]_{i,j\geqslant 1}$. Recall that for any $1 < p < \infty$, T extends to a contraction $S^p \to S^p$ on the Schatten space S^p. Assume that each t_{ij} is real and that there exist $\delta > 0$ such that $-1+\delta \leqslant t_{ij} \leqslant 1$ for any $i,j \geqslant 1$. It was observed in [68] that in this case, $T\colon S^p \to S^p$ is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left(0, \frac{\pi}{2}\right)$, and hence satisfies a square function estimate $\|x\|_{T,1} \lesssim \|x\|$ for $x \in S^p$.

Here is a (brief) alternative proof of this result, using dilations. Results from [68] ensure that it suffices to show that $I - T\colon S^p \to S^p$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{\pi}{2}$. Further arguing as in the proof of Corollary 4.3 of Chapter I, we may reduce to the case when T is unital and completely positive. According to Theorem 4.2 of Chapter I, $T\colon S^p \to S^p$ admits a strict (hence a loose) noncommutative dilation in this case. By Proposition 6.2, this implies that the realisation of $I - T$ on S^p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. Applying [52, Theorem 5.6], to $T_t = e^{-t}e^{tT}$, we find that the realisation of $I-T$ on S^p is R-sectorial of R-type $< \frac{\pi}{2}$. By [56, Proposition 5.1], these two results imply that on S^p, $I - T$ actually admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{\pi}{2}$, as expected.

We refer the reader to Chapter 1 for more examples of operators with a noncommutative strict dilation, and to the next chapter for more about square functions associated to Ritt operators on noncommutative L^p-spaces.

Chapter III

Square functions for Ritt operators on noncommutative L^p-spaces

1 Introduction

Let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace. For any $1 \leqslant p < \infty$, we let $L^p(M)$ denote the associated (noncommutative) L^p-space. Let T a bounded operator on $L^p(M)$. Consider the following 'square function'

$$\|x\|_{T,1} = \inf\left\{\left\|\bigg(\sum_{k=1}^{+\infty}|u_k|^2\bigg)^{\frac{1}{2}}\right\|_{L^p} + \left\|\bigg(\sum_{k=1}^{+\infty}|v_k^*|^2\bigg)^{\frac{1}{2}}\right\|_{L^p} :\ u_k + v_k = k^{\frac{1}{2}}\big(T^k(x) - T^{k-1}(x)\big) \text{ for any } k\right\} \tag{1.1}$$

if $1 < p \leqslant 2$ and

$$\|x\|_{T,1} = \max\left\{\left\|\bigg(\sum_{k=1}^{+\infty}k\left|T^k(x) - T^{k-1}(x)\right|^2\bigg)^{\frac{1}{2}}\right\|_{L^p}, \left\|\bigg(\sum_{k=1}^{+\infty}k\left|\Big(T^k(x) - T^{k-1}(x)\Big)^*\right|^2\bigg)^{\frac{1}{2}}\right\|_{L^p}\right\} \tag{1.2}$$

if $2 \leqslant p < \infty$, defined for any $x \in L^p(M)$. Such quantities were introduced in [68] and studied in this paper and in Chapter 2. Similar square functions for continuous semigroups played a key role in the recent development of H^∞-calculus and its applications. See in particular the paper [52], the survey [67] and the references therein.

For any $\gamma \in \left]0, \frac{\pi}{2}\right[$, let B_γ be the interior of the convex hull of 1 and the disc $D(0, \sin\gamma)$. Suppose $1 < p < \infty$. Let T be a Ritt operator with $\operatorname{Ran}(I - T)$ dense in $L^p(M)$ which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$, i.e. there exists an angle $\gamma \in \left]0, \frac{\pi}{2}\right[$ and a positive constant K such that $\|\varphi(T)\|_{L^p(M) \to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$ for any complex polynomial φ. A result of [68] essentially says that

$$\|x\|_{L^p(M)} \approx \|x\|_{T,1}, \qquad x \in L^p(M) \tag{1.3}$$

(see also Remark 6.4 of Chapter 2). Now, consider the following 'column and row square functions'

$$\|x\|_{T,c,1} = \bigg\| \Big(\sum_{k=1}^{+\infty} k \left| T^k(x) - T^{k-1}(x) \right|^2 \Big)^{\frac{1}{2}} \bigg\|_{L^p} \text{ and } \|x\|_{T,r,1} = \bigg\| \Big(\sum_{k=1}^{+\infty} k \left| \big(T^k(x) - T^{k-1}(x) \big)^* \right|^2 \Big)^{\frac{1}{2}} \bigg\|_{L^p} \tag{1.4}$$

defined for any $x \in L^p(M)$. Assume $1 < p < 2$. In this context, if $x \in L^p(M)$, it is natural to search sufficient conditions to find a decomposition $x = x_1 + x_2$ such that $\|x_1\|_{T,c,1}$ and $\|x_2\|_{T,r,1}$ are finite. The first main result of this paper is the next theorem. It strengthens the above equivalence (1.3) in the case where T actually admits a *completely* bounded $H^\infty(B_\gamma)$ functional calculus, i.e. there exists a positive constant K such that $\|\varphi(T)\|_{cb,L^p(M)\to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$ for any complex polynomial φ.

Theorem 1.1 *Suppose $1 < p < 2$. Let T be a Ritt operator on $L^p(M)$ with $\mathrm{Ran}(I - T)$ dense in $L^p(M)$. Assume that T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$. Then we have*

$$\|x\|_{L^p(M)} \approx \inf\Big\{ \|x_1\|_{T,c,1} + \|x_2\|_{T,r,1} \ : \ x = x_1 + x_2 \Big\}, \qquad x \in L^p(M).$$

In this context, it is natural to compare the both quantities of (1.4). The second principal result of this paper is the following theorem. It says that in general, 'column and row square functions' are not equivalent.

Theorem 1.2 *Suppose $1 < p \neq 2 < \infty$. Then there exists a Ritt operator T on the Schatten space S^p, with $\mathrm{Ran}(I - T)$ dense in S^p, which admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$ such that*

$$\sup\left\{ \frac{\|x\|_{T,c,1}}{\|x\|_{T,r,1}} \ : \ x \in S^p \right\} = \infty \textit{ if } 2 < p < \infty \textit{ and } \sup\left\{ \frac{\|x\|_{T,r,1}}{\|x\|_{T,c,1}} \ : \ x \in S^p \right\} = \infty \textit{ if } 1 < p < 2. \tag{1.5}$$

Moreover, the same result holds with $\|\cdot\|_{T,c,1}$ and $\|\cdot\|_{T,r,1}$ switched.

For a Ritt operator admitting a completely bounded $H^\infty(B_\gamma)$ functional calculus, it also seems interesting, in view of the equivalence (1.3), to compare these both quantities with the usual norm $\|\cdot\|_{L^p(M)}$. If T is a Ritt operator with $\mathrm{Ran}(I - T)$ dense in $L^p(M)$ which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$, the equivalence (1.3) implies that

$$\|x\|_{L^p(M)} \lesssim \|x\|_{T,c,1} \qquad \text{and} \qquad \|x\|_{L^p(M)} \lesssim \|x\|_{T,r,1}$$

if $1 < p \leqslant 2$ and

$$\|x\|_{T,c,1} \lesssim \|x\|_{L^p(M)} \qquad \text{and} \qquad \|x\|_{T,r,1} \lesssim \|x\|_{L^p(M)}$$

if $2 \leqslant p < \infty$, for any $x \in L^p(M)$. The last main result of this paper is that except if $p = 2$, these estimates cannot be reversed:

Theorem 1.3 *Suppose that* $2 < p < \infty$ *(resp.* $1 < p < 2$*). There exists a Ritt operator* T *on the Schatten space* S^p*, with* $\mathrm{Ran}(I - T)$ *dense in* S^p*, which admits a completely bounded* $H^\infty(B_\gamma)$ *functional calculus with* $\gamma \in \left]0, \frac{\pi}{2}\right[$ *such that*

$$\sup\left\{\frac{\|x\|_{S^p}}{\|x\|_{T,c,1}} \ : \ x \in S^p\right\} = \infty \quad \left(\textit{resp.} \ \sup\left\{\frac{\|x\|_{T,c,1}}{\|x\|_{S^p}} \ : \ x \in S^p\right\} = \infty\right).$$

Moreover, the same result holds with $\|\cdot\|_{T,c,1}$ *replaced by* $\|\cdot\|_{T,r,1}$*.*

The paper is organized as follows. Section 2 gives a brief presentation of noncommutative L^p-spaces and Ritt operators and we introduce the notions of Col-Ritt and Row-Ritt operators and completely bounded $H^\infty(B_\gamma)$ functional calculus which are relevant to our paper. The next section 3 mostly contains preliminary results concerning Col-Ritt and Row-Ritt operators. Section 4 is devoted to prove Theorems 1.2 and 1.3. In section 5, we present a proof of Theorem 1.1. We end this section by giving some natural examples to which this result can be applied.

In the above presentation and later on in the paper we will use $\lesssim$ to indicate an inequality up to a constant which does not depend to the particular element to which it applies. Then $A(x) \approx B(x)$ will mean that we both have $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

2 Background and preliminaries

We start with a few preliminaries on noncommutative L^p-spaces. Let M be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let M_+ be the set of all positive elements of M and let S_+ be the set of all x in M_+ such that $\tau(x) < \infty$. Then let S be the linear span of S_+. For any $1 \leqslant p < \infty$, define

$$\|x\|_{L^p(M)} = \left(\tau(|x|^p)\right)^{\frac{1}{p}}, \qquad x \in S,$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the modulus of x. Then $\left(S, \|\cdot\|_{L^p(M)}\right)$ is a normed space. The corresponding completion is the noncommutative L^p-space associated with (M, τ) and is denoted by $L^p(M)$. By convention, we set $L^\infty(M) = M$, equipped with the operator norm. The elements of $L^p(M)$ can also be described as measurable operators with respect to (M, τ). Further multiplication of measurable operators leads to contractive bilinear maps $L^p(M) \times L^q(M) \to L^r(M)$ for any $1 \leqslant p, q, r \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (noncommutative Hölder's inequality). Using trace duality, we then have $L^p(M)^* = L^{p^*}(M)$ isometrically for any $1 \leqslant p < \infty$. Moreover, complex interpolation yields $L^p(M) = \left[L^\infty(M), L^1(M)\right]_{\frac{1}{p}}$ for any $1 \leqslant p \leqslant \infty$. We refer the reader to [103] for details and complements.

Let $1 \leqslant p < \infty$. If we equip the space $B(\ell^2)$ with the operator norm and the canonical trace Tr, the space $L^p(B(\ell^2))$ identifies to the Schatten-von Neumann class S^p. This is the space of those compact operators x from ℓ^2 into ℓ^2 such that $\|x\|_{S^p} = \big(\mathrm{Tr}\,(x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$. Elements of $B(\ell^2)$ or S^p are regarded as matrices $A = [a_{ij}]_{i,j\geqslant 1}$ in the usual way.

If the von Neumann algebra $B(\ell^2)\overline{\otimes}M$ is equipped with the semifinite normal faithful trace $\mathrm{Tr}\otimes\tau$, the space $L^p\big(B(\ell^2)\overline{\otimes}M\big)$ canonically identifies to a space $S^p\big(L^p(M)\big)$ of matrices with entries in $L^p(M)$. Moreover, under this identification, the algebraic tensor product $S^p \otimes L^p(M)$ is dense in $S^p\big(L^p(M)\big)$. We refer to [99] for more about these spaces and complements.

If $1 \leqslant p < \infty$, we say that a linear map on $L^p(M)$ is completely bounded if $I_{S^p}\otimes T$ extends to a bounded operator on $S^p\big(L^p(M)\big)$. In this case, the completely bounded norm $\|T\|_{cb,L^p(M)\to L^p(M)}$ of T is defined by $\|T\|_{cb,L^p(M)\to L^p(M)} = \big\|I_{S^p}\otimes T\big\|_{S^p(L^p(M))\to S^p(L^p(M))}$. We use the convention to define $\|T\|_{cb,L^p(M)\to L^p(M)}$ by $+\infty$ if T is not completely bounded.

We shall use various ℓ^2-valued noncommutative L^p spaces. We refer to [52, Chapter 2] for more information on these spaces. For any $\sum_{k=1}^n x_k\otimes a_k \in L^p(M)\otimes \ell^2$, we set

$$\Bigg\|\sum_{k=1}^n x_k\otimes a_k\Bigg\|_{L^p(M,\ell^2_c)} = \Bigg\|\bigg(\sum_{i,j=1}^n \langle a_j, a_i\rangle x_i^* x_j\bigg)^{\frac{1}{2}}\Bigg\|_{L^p(M)}.$$

We have for any family $(x_k)_{k\geqslant 1}$ in $L^p(M)$

$$\Bigg\|\sum_{k=1}^n x_k\otimes e_k\Bigg\|_{L^p(M,\ell^2_c)} = \Bigg\|\bigg(\sum_{k=1}^n |x_k|^2\bigg)^{\frac{1}{2}}\Bigg\|_{L^p(M)} = \Bigg\|\sum_{k=1}^n e_{k1}\otimes x_k\Bigg\|_{S^p(L^p(M))}. \tag{2.1}$$

The space $L^p(M,\ell^2_c)$ is the completion of $L^p(M)\otimes \ell^2$ for this norm. It identifies to the space of sequences $(x_k)_{k\geqslant 1}$ in $L^p(M)$ such that $\sum_{k=1}^{+\infty} x_k\otimes e_k$ is convergent for the above norm. We define $L^p(M,\ell^2_r)$ similarly. For any finite family $(x_k)_{1\leqslant k\leqslant n}$ in $L^p(M)$, we have

$$\Bigg\|\sum_{k=1}^n x_k\otimes e_k\Bigg\|_{L^p(M,\ell^2_r)} = \Bigg\|\bigg(\sum_{k=1}^n |x_k^*|^2\bigg)^{\frac{1}{2}}\Bigg\|_{L^p(M)} = \Bigg\|\sum_{k=1}^n e_{1k}\otimes x_k\Bigg\|_{S^p(L^p(M))}.$$

For any $1\leqslant p<\infty$ and for any $x_1,\ldots,x_n \in L^p(M)$, we have

$$\Bigg\|\sum_{k=1}^n x_k\otimes e_k\Bigg\|_{L^p(M,\ell^2_c)} = \sup\left\{\Bigg|\sum_{k=1}^n \langle x_k, y_k\rangle_{L^p(M),L^{p^*}(M)}\Bigg| \ :\ \Bigg\|\sum_{k=1}^n y_k\otimes e_k\Bigg\|_{L^{p^*}(M,\ell^2_r)} \leqslant 1\right\}. \tag{2.2}$$

A similar formula holds for the space $L^p(M,\ell^2_r)$. For simplicity, we write $S^p(\ell^2_c)$ for $L^p\big(B(\ell^2),\ell^2_c\big)$. If $2\leqslant p<\infty$ we define the Banach space $L^p(M,\ell^2_{\mathrm{rad}}) = L^p(M,\ell^2_c)\cap L^p(M,\ell^2_r)$. For any $u\in L^p(M,\ell^2_{\mathrm{rad}})$, we have

$$\|u\|_{L^p(\ell^2_{\mathrm{rad}})} = \max\Big\{\|u\|_{L^p(M,\ell^2_c)}, \|u\|_{L^p(M,\ell^2_r)}\Big\}.$$

If $1\leqslant p\leqslant 2$ we define the Banach space $L^p(M,\ell^2_{\mathrm{rad}}) = L^p(M,\ell^2_c)+L^p(M,\ell^2_r)$. For any $u\in L^p(M,\ell^2_{\mathrm{rad}})$, we have

$$\|u\|_{L^p(M,\ell^2_{\mathrm{rad}})} = \inf\Big\{\|u_1\|_{L^p(M,\ell^2_c)} + \|u_2\|_{L^p(M,\ell^2_r)}\Big\}.$$

where the infimum runs over all possible decompositions $u = u_1 + u_2$ with $u_1 \in L^p(M,\ell^2_c)$ and $u_2 \in L^p(M,\ell^2_r)$. Recall that, if $1 < p < \infty$, we have an isometric identification

$$L^p(M,\ell^2_{\mathrm{rad}})^* = L^{p^*}(M,\ell^2_{\mathrm{rad}}). \tag{2.3}$$

Let X be a Banach space and let $(\varepsilon_k)_{k\geqslant 1}$ be a sequence of independent Rademacher variables on some probability space Ω. Let $\mathrm{Rad}(X) \subset L^2(\Omega;X)$ be the closure of $\mathrm{Span}\{\varepsilon_k \otimes x : k \geqslant 1,\ x \in X\}$ in the Bochner space $L^2(\Omega;X)$. Thus for any finite family $x_1,\ldots,x_n$ in X, we have

$$\Big\|\sum_{k=1}^n \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} = \bigg(\int_\Omega \Big\|\sum_{k=1}^n \varepsilon_k(\omega)\, x_k\Big\|_X^2 \,d\omega\bigg)^{\frac{1}{2}}.$$

If $1 \leqslant p < \infty$, the noncommutative Khintchine's inequalities (see [72] and [103]) implies

$$\mathrm{Rad}\big(L^p(M)\big) \approx L^p(M,\ell^2_{\mathrm{rad}}). \tag{2.4}$$

We say that a set $\mathcal{F} \subset B(X)$ is R-bounded if there is a constant $C \geqslant 0$ such that for any finite families $T_1,\ldots,T_n$ in $\mathcal{F}$, and $x_1,\ldots,x_n$ in X, we have

$$\Big\|\sum_{k=1}^n \varepsilon_k \otimes T_k(x_k)\Big\|_{\mathrm{Rad}(X)} \leqslant C\Big\|\sum_{k=1}^n \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $R(\mathcal{F})$ denote the smallest possible C, which is called the R-bound of $\mathcal{F}$. R-boundedness was introduced in [8] and then developed in the fundamental paper [22]. We refer to the latter paper and to [60, Section 2] for a detailed presentation.

On noncommutative L^p-spaces, it will be convenient to consider two naturals variants of this notion, introduced in [52, Chapter 4]. Let $1 < p < \infty$. A subset $\mathcal{F}$ of $B\big(L^p(M)\big)$ is Col-bounded (resp. Row-bounded) if there exists a constant $C \geqslant 0$ such that for any finite families $T_1,\ldots,T_n$ in $\mathcal{F}$, and $x_1,\ldots,x_n$ in $L^p(M)$, we have

$$\bigg\|\bigg(\sum_{k=1}^n |T_k(x_k)|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^p(M)} \leqslant C\bigg\|\bigg(\sum_{k=1}^n |x_k|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^p(M)} \tag{2.5}$$

$$\left(\text{resp. } \bigg\|\bigg(\sum_{k=1}^n |T_k(x_k)^*|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^p(M)} \leqslant C\bigg\|\bigg(\sum_{k=1}^n |x_k^*|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^p(M)}\right). \tag{2.6}$$

The least constant C satisfying (2.5) will be denoted by $\mathrm{Col}(\mathcal{F})$. It follows from (2.4) that if a subset

$\mathcal{F}$ of $B(L^p(M))$ is both Col-bounded and Row-bounded, then it is Rad-bounded.

Note that contrary to the case of R-boundedness, a singleton $\{T\}$ is not automatically Col-bounded or Row-bounded. Indeed, $\{T\}$ is Col-bounded (resp. Row-bounded) if and only if $T \otimes I_{\ell^2}$ extends to a bounded operator on $L^p(M, \ell^2_c)$ (resp. $L^p(M, \ell^2_r)$). And it turns out that if $1 < p \neq 2 < \infty$, according to [52, Example 4.1], there exists a bounded operator T on S^p such that $T \otimes I_{\ell^2}$ does not extend to a bounded operator on $S^p(\ell^2_c)$. Moreover, $T \otimes I_{\ell^2}$ extends to a bounded operator on $S^p(\ell^2_r)$. Then, we also deduce that there are sets $\mathcal{F}$ which are Rad-bounded and Col-bounded without being Row-bounded. Similarly, one may find sets which are Rad-bounded and Row-bounded without being Col-bounded, or which are Rad-bounded without being either Row-bounded or Col-bounded.

We turn to Ritt operators, the key class of this paper, and recall some of their main features. Details and complements can be found in Chapter 2 and in [15], [16], [68], [73], [77], [80] and [116]. Let X be a Banach space. We say that an operator $T \in B(X)$ is a Ritt operator if the two sets

$$\{T^n \,:\, n \geqslant 0\} \qquad \text{and} \qquad \{n(T^n - T^{n-1}) \,:\, n \geqslant 1\} \tag{2.7}$$

are bounded. This is equivalent to the spectral inclusion

$$\sigma(T) \subset \overline{\mathbb{D}} \tag{2.8}$$

and the boundedness of the set

$$\{(\lambda - 1)R(\lambda, T) \,:\, |\lambda| > 1\} \tag{2.9}$$

where $R(\lambda, T) = (\lambda I - T)^{-1}$ denotes the resolvent operator and $\mathbb{D}$ denotes the open unit disc centered at 0. Likewise we say that T is an R-Ritt operator if the two sets in (2.7) are R-bounded. This is equivalent to the inclusion (2.8) and the R-boundedness of the set (2.9).

Let T is a Ritt operator. The boundedness of (2.9) implies the existence of a constant $K \geqslant 0$ such that $|\lambda - 1| \|R(\lambda, T)\|_{X \to X} \leqslant K$ whenever $\Re(\lambda) > 1$. This means that $I - T$ is a sectorial operator. Thus for any $\alpha > 0$, one can consider the fractional power $(I - T)^\alpha$. We refer to [45, Chapter 3], [60] and [74] for various definitions of these (bounded) operators and their basic properties.

We will use the following two naturals variants of the notion of R-Ritt operator.

Definition 2.1 *Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. We say that T is a Col-Ritt (resp. Row-Ritt) operator if the two sets (2.7) are Col-bounded (resp. Row-bounded).*

Remark 2.2 *Assume that $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. Using (2.2), it is easy to see that T is Col-Ritt if and only if T^* is Row-Ritt on $L^{p^*}(M)$.*

We let $\mathcal{P}$ denote the algebra of all complex polynomials. Let T be a bounded operator on a Banach space X. Let $\gamma \in \left]0, \frac{\pi}{2}\right[$. Accordingly with [68], we say that T has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if there exists a constant $K \geqslant 1$ such that

$$\|\varphi(T)\|_{X\to X} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$$

for any $\varphi \in \mathcal{P}$. Naturally, we let:

Definition 2.3 *Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. Let $\gamma \in \left]0, \frac{\pi}{2}\right[$. We say that T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus if T is completely bounded and if $I_{S^p} \otimes T$ admits a bounded $H^\infty(B_\gamma)$ functional calculus on $S^p(L^p(M))$.*

Let T be a bounded operator on $L^p(M)$ and $\gamma \in \left]0, \frac{\pi}{2}\right[$. Note that T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus if and only if there exists a constant $K \geqslant 1$ such that

$$\|\varphi(T)\|_{cb, L^p(M)\to L^p(M)} \leqslant K\|\varphi\|_{H^\infty(B_\gamma)}$$

for any $\varphi \in \mathcal{P}$.

3 Results related to Col-Ritt or Row-Ritt operators

In the subsequent sections, we need some preliminary results on Col-Ritt or Row-Ritt operators that we present here. Some of them are analogues of existing results in the context of R-Ritt operators, for which we will omit proofs.

We start with a variant of Proposition II.2.8 suitable with our context. The proof is similar, using [52, Lemma 4.2] instead of Lemma II.2.1.

Proposition 3.1 *Suppose $1 < p < \infty$. Let T be a Col-Ritt operator on $L^p(M)$. For any $\alpha > 0$, the set*

$$\left\{ n^\alpha (\varrho T)^{n-1}(I - \varrho T)^\alpha : \ n \geqslant 1, \ \varrho \in]0,1] \right\}$$

is Col-bounded. Moreover, a similar result holds for Row-Ritt operators.

Moreover, we need the following result [68].

Theorem 3.2 *Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$ with a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$. Then T is R-Ritt.*

In the next statement, we establish a variant of the above result.

Theorem 3.3 *Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. Assume that T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$. Then the operator T is both Col-Ritt and Row-Ritt.*

Proof : We will only show the 'column' result, the proof for the 'row' one being the same. We wish to show that the sets

$$\mathcal{F} = \{T^m \,:\, m \geqslant 0\} \qquad \text{and} \qquad \mathcal{G} = \{m(T^m - T^{m-1}) \,:\, m \geqslant 1\}$$

are Col-bounded. We consider the operator $I \otimes T$ on the noncommutative L^p-space $S^p(L^p(M))$. Then, applying Theorem 3.2, we obtain that the sets

$$\mathcal{T} = \{I_{S^p} \otimes T^m \,:\, m \geqslant 0\} \qquad \text{and} \qquad \mathcal{K} = \{mI_{S^p} \otimes (T^m - T^{m-1}) \,:\, m \geqslant 1\}$$

are Rad-bounded. Now consider $x_1, \ldots, x_n$ in $L^p(M)$ and $T_1, \ldots, T_n$ in $\mathcal{F}$. For any finite sequence $(\varepsilon_k)_{1 \leqslant k \leqslant n}$ valued in $\{-1, 1\}$, we have

$$\begin{aligned} \Bigg\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^p(M)} &= \Bigg\| \Big(\sum_{k=1}^{n} (\varepsilon_k x_k)^* (\varepsilon_k x_k) \Big)^{\frac{1}{2}} \Bigg\|_{L^p(M)} \\ &= \Bigg\| \sum_{k=1}^{n} \varepsilon_k e_{k1} \otimes x_k \Bigg\|_{S^p(L^p(M))}. \end{aligned}$$

Then passing to the average over all possible choices of $\varepsilon_k = \pm 1$, we obtain that

$$\Bigg\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^p(M)} = \Bigg\| \sum_{k=1}^{n} \varepsilon_k \otimes e_{k1} \otimes x_k \Bigg\|_{\mathrm{Rad}(S^p(L^p(M)))}.$$

By a similar computation, we have

$$\Bigg\| \Big(\sum_{k=1}^{n} |T_k(x_k)|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^p(M)} = \Bigg\| \sum_{k=1}^{n} \varepsilon_k \otimes (I_{S^p} \otimes T_k)(e_{k1} \otimes x_k) \Bigg\|_{\mathrm{Rad}(S^p(L^p(M)))}.$$

It follows that

$$\Bigg\| \Big(\sum_{k=1}^{n} |T_k(x_k)|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^p(M)} \leqslant \mathrm{Rad}(\mathcal{T}) \Bigg\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^p(M)}.$$

This concludes the proof of Col-boundedness of $\mathcal{F}$ with $\mathrm{Col}(\mathcal{F}) \leqslant \mathrm{Rad}(\mathcal{T})$. The proof for the set $\mathcal{G}$ is identical. ■

Remark 3.4 *Suppose $1 < p \neq 2 < \infty$. The complete boundedness assumption in Theorem 3.3 cannot be replaced by a boundedness assumption.*

Proof : We have already recalled that, there exists a bounded operator T on S^p such that $\{T\}$ is not Col-bounded. Let us fix $\gamma \in \left]0, \frac{\pi}{2}\right[$. We may clearly assume that $\sigma(T)$ is included in the open set B_γ. Using the Dunford calculus, it is easy to prove that T is a Ritt operator which admits a bounded

$H^\infty(B_\gamma)$ functional calculus. The set $\{T\}$ is not Col-bounded. Hence T cannot be Col-Ritt. ■

Now, we give a precise definition of 'square functions' which clarifies (1.1), (1.2) and (1.4) and a few comments. Let T a Ritt operator on $L^p(M)$. For any $\alpha > 0$, let us consider

$$x_k = k^{\alpha - \frac{1}{2}} T^{k-1}(I-T)^\alpha(x)$$

for any $k \geqslant 1$. If the sequence belongs to the space $L^p(M, \ell^2_c)$, then $\|x\|_{T,c,\alpha}$ is defined as the norm of $(x_k)_{k\geqslant 1}$ in that space. Otherwise, we set $\|x\|_{T,c,\alpha} = \infty$. In particular, $\|x\|_{T,c,\alpha}$ can be infinite. We define the quantities $\|x\|_{T,r,\alpha}$ by the same way. The quantities $\|x\|_{T,\alpha}$ are defined similarly in Chapter 2, using the space $L^p(M, \ell^2_{\mathrm{rad}})$ instead of $L^p(M, \ell^2_c)$.

Finally, note that, if $2 \leqslant p < \infty$, we have

$$\|x\|_{T,\alpha} = \max\big\{\|x\|_{T,c,\alpha}, \|x\|_{T,r,\alpha}\big\}.$$

and if $1 \leqslant p \leqslant 2$, we have

$$\|x\|_{T,\alpha} = \inf\Big\{\|u\|_{L^p(M,\ell^2_c)} + \|v\|_{L^p(M,\ell^2_r)} \,:\, u_k + v_k = k^{\alpha-\frac{1}{2}} T^{k-1}(I-T)^\alpha x \text{ for any integer } k\Big\}.$$

In [68], the following connection between the boundedness of square functions and functional calculus is established.

Theorem 3.5 *Suppose $1 < p < \infty$. Let T be a bounded operator on $L^p(M)$. The following assertions are equivalent.*

1. *The operator T is R-Ritt and T and its adjoint T^* both satisfy uniform estimates*

$$\|x\|_{T,1} \lesssim \|x\|_{L^p(M)} \qquad \text{and} \qquad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}(M)}$$

   for any $x \in L^p(M)$ and $y \in L^{p^*}(M)$.

2. *The operator T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$.*

Recall a special case of the principal result of Chapter 2.

Theorem 3.6 *Let T be an R-Ritt operator on $L^p(M)$ with $1 < p < \infty$. For any $\alpha, \beta > 0$ we have an equivalence*

$$\|x\|_{T,\alpha} \approx \|x\|_{T,\beta}, \qquad x \in L^p(M).$$

We shall now present a variant suitable to our context.

For any integer $n \geqslant 1$, we identify the algebra M_n of all $n \times n$ matrices with the space of linear maps $\ell^2_n \to \ell^2_n$. For any infinite matrix $[c_{ij}]_{i,j\geqslant 1}$, we set

$$\big\|[c_{ij}]\big\|_{\mathrm{reg}} = \sup_{n\geqslant 1} \Big\|\big[|c_{ij}|\big]_{1\leqslant i,j\leqslant n}\Big\|_{B(\ell^2_n)}$$

This is the so-called 'regular norm'. We refer to [94] and [102] for more information on regular norms.

The next proposition will be useful. This result is similar to Proposition 2.6 of Chapter 2.

Proposition 3.7 *Suppose* $1 < p < \infty$. *Let* $[c_{ij}]_{i,j\geqslant 1}$ *be an infinite matrix with* $\left\|[c_{ij}]\right\|_{\mathrm{reg}} < \infty$. *Suppose that* $\{T_{ij} \,:\, i,j \geqslant 1\}$ *is a Col-bounded set of operators on* $L^p(M)$. *Then the linear map*

$$\begin{array}{rccc} [c_{ij}T_{ij}]: & L^p(M,\ell^2_c) & \longrightarrow & L^p(M,\ell^2_c) \\ & \displaystyle\sum_{j=1}^{+\infty} x_j \otimes e_j & \longmapsto & \displaystyle\sum_{i=1}^{+\infty}\left(\sum_{j=1}^{+\infty} c_{ij}T_{ij}(x_j)\right)\otimes e_i \end{array}$$

is well-defined and bounded. Moreover, we have a similar result for Row-bounded sets.

Proof : We shall only prove the 'Col' result. We can assume that $\left\|[c_{ij}]\right\|_{\mathrm{reg}} \leqslant 1$. Let $n \geqslant 1$. By Lemma 2.2 of Chapter 2, we can write $c_{ij} = a_{ij}b_{ij}$ for any $1 \leqslant i,j \leqslant n$ with

$$\sup_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|a_{ij}|^2 \leqslant 1 \qquad \text{and} \qquad \sup_{1\leqslant j\leqslant n}\sum_{i=1}^{n}|b_{ij}|^2 \leqslant 1.$$

Let $x_1,\ldots,x_n \in L^p(M)$ and $y_1,\ldots,y_n \in L^{p^*}(M)$. Since the set $\{T_{ij} \mid i,j \geqslant 1\}$ is Col-bounded, there exists a positive constant C such that

$$\begin{aligned} \left|\sum_{i=1}^{n}\left\langle \sum_{j=1}^{n} c_{ij}T_{ij}(x_j), y_i\right\rangle_{L^p(M),L^{p^*}(M)}\right| &= \left|\sum_{i,j=1}^{n}\langle a_{ij}b_{ij}T_{ij}(x_j),y_i\rangle_{L^p(M),L^{p^*}(M)}\right| \\ &= \left|\sum_{i,j=1}^{n}\langle T_{ij}(b_{ij}x_j),a_{ij}y_i\rangle_{L^p(M),L^{p^*}(M)}\right| \\ &\leqslant \left\|\left(\sum_{i,j=1}^{n}|T_{ij}(b_{ij}x_j)|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)}\left\|\left(\sum_{i,j=1}^{n}|(a_{ij}y_i)^*|^2\right)^{\frac{1}{2}}\right\|_{L^{p^*}(M)} \\ &\leqslant C\left\|\left(\sum_{i,j=1}^{n}|b_{ij}x_j|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)}\left\|\left(\sum_{i,j=1}^{n}|a_{ij}y_i^*|^2\right)^{\frac{1}{2}}\right\|_{L^{p^*}(M)}. \end{aligned}$$

Now, we have

$$\sum_{i,j=1}^{n}|b_{ij}x_j|^2 = \sum_{j=1}^{n}|x_j|^2\left(\sum_{i=1}^{n}|b_{ij}|^2\right) \leqslant \sum_{j=1}^{n}|x_j|^2.$$

Similarly, we have

$$\sum_{i,j=1}^{n}|a_{ij}y_i^*|^2 \leqslant \sum_{i=1}^{n}|y_i^*|^2.$$

Consequently

$$\left|\sum_{i=1}^{n}\left\langle \sum_{j=1}^{n} c_{ij}T_{ij}(x_j), y_i\right\rangle_{L^p(M),L^{p^*}(M)}\right| \leqslant C\left\|\left(\sum_{j=1}^{n}|x_j|^2\right)^{\frac{1}{2}}\right\|_{L^p(M)}\left\|\left(\sum_{i=1}^{n}|y_i^*|^2\right)^{\frac{1}{2}}\right\|_{L^{p^*}(M)}.$$

Taking the supremum over all $y_1,\ldots,y_n \in L^{p^*}(M)$ such that $\|(\sum_{i=1}^{n}|y_i^*|^2)^{\frac{1}{2}}\|_{L^{p^*}(M)} \leqslant 1$, we obtain

$$\left\|\sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}T_{ij}(x_j)\right)\otimes e_i\right\|_{L^p(M,\ell^2_c)} \leqslant C\left\|\sum_{j=1}^{n} x_j\otimes e_j\right\|_{L^p(M,\ell^2_c)}$$

by (2.2). We conclude with [52, Corollary 2.12]. ■

Now, we state a result which allows to estimate square functions $\|\cdot\|_{T,c,\alpha}$ and $\|\cdot\|_{T,r,\alpha}$ by means of approximation processes, whose proof is similar to Lemma 3.2 of Chapter 2.

Lemma 3.8 *Suppose $1<p<\infty$. Assume that T is a Col-Ritt operator on $L^p(M)$. Let $\alpha>0$.*

1. *Let V be an operator on $L^p(M)$ such that $TV=VT$ with $\{V\}$ Col-bounded. Then, for any $x\in L^p(M)$, we have*
$$\|V(x)\|_{T,c,\alpha} \leqslant \mathrm{Col}(\{V\})\|x\|_{T,c,\alpha}.$$
2. *Let $\nu \geqslant \alpha+1$ be an integer and let $x \in \mathrm{Ran}\big((I-T)^{\nu}\big)$. Then*
$$\|x\|_{\varrho T,c,\alpha} \xrightarrow[\varrho\to 1^-]{} \|x\|_{T,c,\alpha}.$$

Moreover, the same result holds with $\|\cdot\|_{T,c,\alpha}$ replaced by $\|\cdot\|_{T,r,\alpha}$ for Row-Ritt operators.

Now we state an equivalence result in our context similar to Theorem 3.6.

Theorem 3.9 *Let T be a bounded operator on $L^p(M)$ with $1<p<\infty$. Let $\alpha,\beta>0$.*

1. *If T is Col-Ritt, we have an equivalence*
$$\|x\|_{T,c,\alpha} \approx \|x\|_{T,c,\beta}, \qquad x\in L^p(M).$$
2. *If T is Row-Ritt, we have an equivalence*
$$\|x\|_{T,r,\alpha} \approx \|x\|_{T,r,\beta}, \qquad x\in L^p(M).$$

Proof : The proof is similar to the one of Theorem 3.3 of Chapter 2, using Proposition 3.1, Proposition 3.7, Lemma 3.8 and [52, Corollary 2.12]. ■

4 Comparison between squares functions and the usual norm

We aim at showing Theorem 1.3. We will provide an example on the Schatten space S^p. This example also prove that in general, row and column square functions are not equivalent (Theorem 4.3).

Let a a bounded operator on ℓ^2. Assume $1 < p < \infty$. We let $\mathcal{L}_a \colon S^p \to S^p$ the left multiplication by a on S^p defined by $\mathcal{L}_a(x) = ax$ and we denote $\mathcal{R}_a \colon S^p \to S^p$ the right multiplication. It is clear that $\mathcal{L}_a^*$ and $\mathcal{R}_a^*$ are the right multiplication and the left multiplication by a on S^{p^*}. Note that, by [52, Proposition 8.4 (4)], if $I - a$ has dense range then $\mathrm{Ran}(I - \mathcal{L}_a)$ is dense in S^p. The next statement gives a link between properties of a and its associated multiplication operators.

Proposition 4.1 *Suppose $1 < p < \infty$. Assume that a is a bounded operator on ℓ^2.*

1. *If a is a Ritt operator then the left multiplication $\mathcal{L}_a$ is a Ritt operator on S^p.*
2. *Let $\gamma \in \left]0, \frac{\pi}{2}\right[$. Then $\mathcal{L}_a$ has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if a has one. In that case, $\mathcal{L}_a$ actually has a completely bounded $H^\infty(B_\gamma)$ functional calculus.*

Moreover, we have a similar result for right multiplication.

Proof : We have $\sigma(\mathcal{L}_a) \subset \sigma(a)$. Moreover, if $\lambda \in \rho(a)$ we have $R(\lambda, \mathcal{L}_a) = \mathcal{L}_{R(\lambda,a)}$. The first assertion clearly follows. The statement (2) is a straightforward consequence of

$$I_{S^p} \otimes \mathcal{L}_a = \mathcal{L}_{I_{\ell^2} \otimes a} \quad \text{and} \quad f(\mathcal{L}_a) = \mathcal{L}_{f(a)}, \qquad f \in \mathcal{P}.$$

The proof of the 'right' result is identical. ■

We denote by $(e_k)_{k \geqslant 1}$ the canonical basis of ℓ^2. Now, for any integer $k \geqslant 1$, we fix $a_k = 1 - \frac{1}{2^k}$. We consider the selfadjoint bounded diagonal operator a on ℓ^2 defined by

$$a\left(\sum_{k=1}^{+\infty} x_k e_k\right) = \sum_{k=1}^{+\infty} a_k x_k e_k.$$

It follows from the Spectral Theorem for normal operators, that the operator a admits a bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in \left]0, \frac{\pi}{2}\right[$. Thus $\mathcal{L}_a$ and $\mathcal{R}_a$ admit a completely bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in \left]0, \frac{\pi}{2}\right[$ (hence $\mathcal{L}_a$ and $\mathcal{R}_a$ are Ritt operators).

Lemma 4.2 *Assume that $2 \leqslant p < \infty$. We have*

$$\|x\|_{\mathcal{L}_a,c,1} \approx \|x\|_{S^p} \quad \textit{and} \quad \|x\|_{\mathcal{R}_a,r,1} \approx \|x\|_{S^p}, \qquad x \in S^p. \tag{4.1}$$

Proof : We will only show the result for the operator $\mathcal{L}_a$, the proof for $\mathcal{R}_a$ being the same. For any $x \in S^p$ and any $\varrho \in]0,1[$, we have

$$k\big((\varrho\mathcal{L}_a)^{k-1}(I - \varrho\mathcal{L}_a)(x)\big)^*\big((\varrho\mathcal{L}_a)^{k-1}(I - \varrho\mathcal{L}_a)(x)\big) = k\big((\varrho a)^{k-1}(I - \varrho a)x\big)^*\big((\varrho a)^{k-1}(I - \varrho a)x\big)$$

$$\begin{aligned} &= kx^*(I-\varrho a)(\varrho a)^{2(k-1)}(I-\varrho a)x \\ &= kx^*(I-\varrho\mathfrak{L}_a)^2(\varrho\mathfrak{L}_a)^{2(k-1)}(x). \end{aligned}$$

Now, for any $z \in \mathbb{D}$, we have

$$\sum_{k=1}^{+\infty} kz^{k-1} = (1-z)^{-2}. \tag{4.2}$$

Since the operator $\mathfrak{L}_a$ is a contraction, we deduce that, for every $\varrho \in]0,1[$, the operator $I-(\varrho\mathfrak{L}_a)^2$ is invertible and that we have

$$\sum_{k=1}^{+\infty} k(\varrho\mathfrak{L}_a)^{2(k-1)} = \big(I-(\varrho\mathfrak{L}_a)^2\big)^{-2}, \tag{4.3}$$

the series being absolutely convergent. Then we deduce that the series

$$\sum_{k=1}^{+\infty} k\big((\varrho\mathfrak{L}_a)^{k-1}(I-\varrho\mathfrak{L}_a)(x)\big)^*\big((\varrho\mathfrak{L}_a)^{k-1}(I-\varrho\mathfrak{L}_a)(x)\big)$$

is convergent in the Banach space $S^{\frac{p}{2}}$ and that

$$\begin{aligned} \sum_{k=1}^{+\infty} k\big((\varrho\mathfrak{L}_a)^{k-1}(I-\varrho\mathfrak{L}_a)(x)\big)^*\big((\varrho\mathfrak{L}_a)^{k-1}(I-\varrho\mathfrak{L}_a)(x)\big) &= x^*(I-\varrho\mathfrak{L}_a)^2\big(I-(\varrho\mathfrak{L}_a)^2\big)^{-2}x \\ &= x^*(I+\varrho a)^{-2}x. \end{aligned}$$

We deduce that

$$\begin{aligned} \|x\|_{\varrho\mathfrak{L}_a,c,1} &= \Big\|\Big(x^*(I+\varrho a)^{-2}x\Big)^{\frac{1}{2}}\Big\|_{S^p} \\ &= \big\|(I+\varrho a)^{-1}x\big\|_{S^p}. \end{aligned}$$

Then, for any $x \in S^p$, we obtain the estimate

$$\begin{aligned} \|x\|_{\varrho\mathfrak{L}_a,c,1} &\leqslant \big\|(I+\varrho a)^{-1}\big\|_{B(\ell^2)}\|x\|_{S^p} \\ &\leqslant \|x\|_{S^p}. \end{aligned}$$

By a similar computation, for any $x \in S^p$, we have

$$\frac{1}{2}\|x\|_{S^p} \leqslant \|x\|_{\varrho\mathfrak{L}_a,c,1}.$$

Applying Lemma 3.8 (2), we deduce an equivalence

$$\frac{1}{2}\|x\|_{S^p} \leqslant \|x\|_{\mathfrak{L}_a,c,1} \leqslant \|x\|_{S^p}, \qquad x \in \mathrm{Ran}\big((I-\mathfrak{L}_a)^2\big).$$

For any integer $n \geqslant 1$, we let d_n the bounded diagonal operator on ℓ^2 defined by the matrix $\operatorname{diag}(1,\dots,1,0,\dots)$. It is not difficult to see that, for any integer $n \geqslant 1$, the range of $\mathcal{L}_{d_n}$ is a subspace of $\operatorname{Ran}\big((I-\mathcal{L}_a)^2\big)$. Hence we actually have

$$\frac{1}{2}\big\|\mathcal{L}_{d_n}(x)\big\|_{S^p} \leqslant \big\|\mathcal{L}_{d_n}(x)\big\|_{\mathcal{L}_a,c,1} \leqslant \big\|\mathcal{L}_{d_n}(x)\big\|_{S^p}, \qquad x \in S^p, \quad n \geqslant 1.$$

Then, on the one hand, we obtain

$$\big\|\mathcal{L}_{d_n}(x)\big\|_{\mathcal{L}_a,c,1} \leqslant \|x\|_{S^p}, \qquad x \in S^p, \quad n \geqslant 1.$$

By [52, Corollary 2.12] and (2.1), this latter inequality is equivalent to

$$\Bigg\|\sum_{k=1}^{l} e_{k1} \otimes k^{\frac{1}{2}}\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)\big(\mathcal{L}_{d_n}(x)\big)\Bigg\|_{S^p(S^p)} \lesssim \|x\|_{S^p}, \qquad x \in S^p, \quad n \geqslant 1, \quad l \geqslant 1.$$

Passing to the limit in the above inequality and using again [52, Corollary 2.12], we obtain that

$$\|x\|_{\mathcal{L}_a,c,1} \leqslant \|x\|_{S^p}, \qquad x \in S^p.$$

Note, in particular that, for any $x \in S^p$, we have $\|x\|_{\mathcal{L}_a,c,1} < \infty$. On the other hand, note that, for any integer $n \geqslant 1$, the operators $\mathcal{L}_a$ and $\mathcal{L}_{d_n}$ commute. Hence, for any $x \in S^p$ and any integer $n \geqslant 1$, we have

$$\begin{aligned}
\|\mathcal{L}_{d_n}(x)\|_{S^p} &\lesssim \big\|\mathcal{L}_{d_n}(x)\big\|_{\mathcal{L}_a,c,1} \\
&= \Bigg\|\sum_{k=1}^{+\infty} e_{k1} \otimes k^{\frac{1}{2}}\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)\big(\mathcal{L}_{d_n}(x)\big)\Bigg\|_{S^p(S^p)} \\
&= \Bigg\|(I_{S^p} \otimes \mathcal{L}_{d_n})\Bigg(\sum_{k=1}^{+\infty} e_{k1} \otimes k^{\frac{1}{2}}\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\Bigg)\Bigg\|_{S^p(S^p)}.
\end{aligned}$$

Letting n to the infinity, we deduce that

$$\|x\|_{S^p} \lesssim \|x\|_{\mathcal{L}_a,c,1}, \qquad x \in S^p.$$

The proof is complete. ■

Theorem 4.3 *Let $\alpha > 0$. Assume that $2 < p < \infty$. Then*

$$\sup\left\{\frac{\|x\|_{\mathcal{L}_a,c,\alpha}}{\|x\|_{\mathcal{L}_a,r,\alpha}} \ :\ x \in S^p\right\} = \infty \qquad \textit{and} \qquad \sup\left\{\frac{\|x\|_{\mathcal{R}_a,r,\alpha}}{\|x\|_{\mathcal{R}_a,c,\alpha}} \ :\ x \in S^p\right\} = \infty. \tag{4.4}$$

Assume that $1 < p < 2$. *Then*

$$\sup\left\{\frac{\|x\|_{\mathcal{L}_a,r,\alpha}}{\|x\|_{\mathcal{L}_a,c,\alpha}} \ : \ x \in S^p\right\} = \infty \qquad \textit{and} \qquad \sup\left\{\frac{\|x\|_{\mathcal{R}_a,c,\alpha}}{\|x\|_{\mathcal{R}_a,r,\alpha}} \ : \ x \in S^p\right\} = \infty. \tag{4.5}$$

Proof : By Theorem 3.9, it suffices to prove the result for one specific real α. Throughout the proof, we will use $\alpha = 1$. We first assume that $2 < p < \infty$. Given an integer $n \geqslant 1$, we consider $e = e_1 + \cdots + e_n \in \ell^2_n$ and $x = \frac{1}{\sqrt{n}} e \otimes e \in S^p$. Clearly, we have

$$xx^* = \sum_{i,j=1}^{n} e_{ij}.$$

Now, we have

$$\begin{aligned}
k\big(\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\big)\big(\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\big)^* &= k\big(a^{k-1}(I-a)x\big)\big(a^{k-1}(I-a)x\big)^* \\
&= ka^{k-1}(I-a)xx^*(I-a)a^{k-1} \\
&= \sum_{i,j=1}^{n} ka^{k-1}(I-a)e_{ij}(I-a)a^{k-1} \\
&= \sum_{i,j=1}^{n} (1-a_i)(1-a_j)k(a_ia_j)^{k-1}e_{ij}.
\end{aligned}$$

Using the equality (4.2), we obtain that the series

$$\sum_{k=1}^{+\infty} k\big(\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\big)\big(\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\big)^*$$

is convergent in $S^{\frac{p}{2}}$ and that

$$\sum_{k=1}^{+\infty} k\big(\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\big)\big(\mathcal{L}_a^{k-1}(I-\mathcal{L}_a)(x)\big)^* = \sum_{i,j=1}^{n} (1-a_i)(1-a_j)(1-a_ia_j)^{-2}e_{ij}.$$

Now, note that

$$(1-a_i)(1-a_j)(1-a_ia_j)^{-2} = \frac{2^{i+j}}{(2^i+2^j-1)^2}.$$

We deduce that

$$\begin{aligned}
\|x\|_{\mathcal{L}_a,r,1} &= \left\|\left(\sum_{i,j=1}^{n} \frac{2^{i+j}}{(2^i+2^j-1)^2}e_{ij}\right)^{\frac{1}{2}}\right\|_{S^p} \\
&= \left\|\sum_{i,j=1}^{n} \frac{2^{i+j}}{(2^i+2^j-1)^2}e_{ij}\right\|_{S^{\frac{p}{2}}}^{\frac{1}{2}}.
\end{aligned}$$

We let $A = \left[\frac{2^{i+j}}{(2^i+2^j-1)^2}\right]_{1\leqslant i,j\leqslant n}$ be the $n\times n$ matrix in the last right member of the above equations. We have

$$\begin{aligned}\|A\|_{S_n^2}^2 &= \sum_{i,j=1}^{n}\left(\frac{2^{i+j}}{(2^i+2^j-1)^2}\right)^2\\ &= \sum_{i,j=1}^{n}\frac{4^{i+j}}{(2^i+2^j-1)^4}.\end{aligned}$$

Moreover, note that

$$\begin{aligned}\frac{4^{i+j}}{(2^i+2^j-1)^4} &\leqslant 16\frac{4^{i+j}}{(2^i+2^j)^4}\\ &= 16\left(\frac{1}{2^{i-j}+2^{j-i}+2}\right)^2\\ &\leqslant \frac{16}{4^{|i-j|}}.\end{aligned}$$

Thus we have

$$\|A\|_{S_n^2}^2 \leqslant 32\left(\sum_{k\in\mathbb{Z}}\frac{1}{4^{|k|}}\right)n \approx n.$$

If $4\leqslant p<\infty$, we obtain

$$\|x\|_{\mathcal{L}_{a,r,1}} = \|A\|_{S_n^{\frac{p}{2}}}^{\frac{1}{2}} \leqslant \|A\|_{S_n^2}^{\frac{1}{2}} \lesssim n^{\frac{1}{4}}.$$

Since $x=\frac{1}{\sqrt{n}}e\otimes e$ is rank one, its norm in S^p does not depend on p, and it is equal to $\frac{1}{\sqrt{n}}\|e\|_{\ell_n^2}^2=\sqrt{n}$. Then, by Lemma 4.2, we have $\|x\|_{\mathcal{L}_{a,c,1}}\approx\sqrt{n}$. We obtain the first equality of (4.4) in that case.

If $2<p\leqslant 4$, we can write $\frac{1}{\frac{p}{2}}=\frac{1-\theta}{1}+\frac{\theta}{2}$ with $0<\theta\leqslant 1$. Then

$$\|x\|_{\mathcal{L}_{a,r,1}}^2 = \|A\|_{S_n^{\frac{p}{2}}} \leqslant \|A\|_{S_n^1}^{1-\theta}\|A\|_{S_n^2}^{\theta}.$$

By construction, we have $A\geqslant 0$, hence we have

$$\begin{aligned}\|A\|_{S_n^1} &= \mathrm{Tr}\left(\sum_{i,j=1}^{n}\frac{2^{i+j}}{(2^i+2^j-1)^2}e_{ij}\right)\\ &= \sum_{i=1}^{n}\frac{4^i}{(2^{i+1}-1)^2} \leqslant \sum_{i=1}^{n}\frac{4^i}{(2^i)^2} = n.\end{aligned}$$

Thus

$$\|x\|_{\mathcal{L}_{a,r,1}}^2 \lesssim n^{1-\theta}n^{\frac{\theta}{2}} = n^{1-\frac{\theta}{2}}.$$

Recall that $\|x\|_{\mathcal{L}_a,c,1} \approx \sqrt{n}$. We obtain that

$$\frac{\|x\|_{\mathcal{L}_a,c,1}}{\|x\|_{\mathcal{L}_a,r,1}} \gtrsim \frac{n^{\frac{1}{2}}}{n^{\frac{1}{2}-\frac{\theta}{4}}} = n^{\frac{\theta}{4}}.$$

Since n was arbitrary and $\theta > 0$, we obtain the first part of (4.4) in this case. Likewise, the above proof has a 'right analog' which proves the second equality of (4.4).

We now turn to the proof of (4.5). We assume that $1 < p < 2$. The second part of (4.4) says

$$\sup\left\{\frac{\|y\|_{\mathcal{L}_a^*,r,1}}{\|y\|_{\mathcal{L}_a^*,c,1}} \ : \ y \in S^{p^*}\right\} = \infty. \tag{4.6}$$

To prove the first equality of (4.5), assume on the contrary that there is a constant $K > 0$ such that for any $x \in S^p$

$$\|x\|_{\mathcal{L}_a,r,1} \leqslant K\|x\|_{\mathcal{L}_a,c,1}. \tag{4.7}$$

We begin by showing a duality relation between $\|\cdot\|_{\mathcal{L}_a^*,c,1}$ and $\|\cdot\|_{\mathcal{L}_a,r,1}$. Let $y \in S^{p^*}$ and $x \in S^p$. For any integer $n \geqslant 1$, recall that d_n is the bounded diagonal operator on ℓ^2 defined by the matrix $\mathrm{diag}(1,\ldots,1,0,\ldots)$. By (4.3), for any $0 < \varrho < 1$ and any integer $n \geqslant 1$, we have

$$\begin{aligned}
\left|\left\langle y, \mathcal{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| &= \left|\left\langle y, \sum_{k=1}^{+\infty} k(\varrho\mathcal{L}_a)^{2(k-1)}(I-(\varrho\mathcal{L}_a)^2)^2\mathcal{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| \\
&= \left|\sum_{k=1}^{+\infty}\left\langle y, k(\varrho\mathcal{L}_a)^{2(k-1)}(I-(\varrho\mathcal{L}_a)^2)^2\mathcal{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| \\
&= \left|\sum_{k=1}^{+\infty}\left\langle k^{\frac{1}{2}}(\varrho\mathcal{L}_a^*)^{k-1}(I-\varrho\mathcal{L}_a^*)(I+\varrho\mathcal{L}_a^*)^2 y, k^{\frac{1}{2}}(\varrho\mathcal{L}_a)^{k-1}(I-\varrho\mathcal{L}_a)\mathcal{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| \\
&\leqslant \left\|\left(k^{\frac{1}{2}}(\varrho\mathcal{L}_a^*)^{k-1}(I-\varrho\mathcal{L}_a^*)(I+\varrho\mathcal{L}_a^*)^2 y\right)_{k\geqslant 1}\right\|_{S^p(\ell^2_c)} \left\|\mathcal{L}_{d_n}(x)\right\|_{\varrho\mathcal{L}_a,r,1}.
\end{aligned}$$

Now, it is easy to see that $\{\mathcal{L}_a^*\}$ is Col-bounded. We infer that

$$\begin{aligned}
\left|\left\langle y, \mathcal{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| &\lesssim \left\|\left(k^{\frac{1}{2}}(\varrho\mathcal{L}_a^*)^{k-1}(I-\varrho\mathcal{L}_a^*)y\right)_{k\geqslant 1}\right\|_{S^p(\ell^2_c)} \left\|\mathcal{L}_{d_n}(x)\right\|_{\varrho\mathcal{L}_a,r,1} \\
&= \|y\|_{\varrho\mathcal{L}_a^*,c,1}\left\|\mathcal{L}_{d_n}(x)\right\|_{\varrho\mathcal{L}_a,r,1}.
\end{aligned}$$

Assume for a while that $y \in \mathrm{Ran}((I-\mathcal{L}_a^*)^2)$. By Lemma 3.8 (2), letting ϱ to 1, we obtain

$$\left|\left\langle y, \mathcal{L}_{d_n}(x)\right\rangle_{S^{p^*},S^p}\right| \lesssim \|y\|_{\mathcal{L}_a^*,c,1}\left\|\mathcal{L}_{d_n}(x)\right\|_{\mathcal{L}_a,r,1}.$$

Letting n to the infinity, we obtain

$$\left|\langle y, x\rangle_{S^{p^*}, S^p}\right| \lesssim \|y\|_{\mathcal{L}_a^*, c, 1} \|x\|_{\mathcal{L}_a, r, 1}.$$

According to (4.7) and the first part of (4.1), we deduce that

$$\begin{aligned}\left|\langle y, x\rangle_{S^{p^*}, S^p}\right| &\lesssim \|y\|_{\mathcal{L}_a^*, c, 1} \|x\|_{\mathcal{L}_a, c, 1} \\ &\lesssim \|y\|_{\mathcal{L}_a^*, c, 1} \|x\|_{S^p}.\end{aligned}$$

By duality, we finally obtain that

$$\|y\|_{S^{p^*}} \lesssim \|y\|_{\mathcal{L}_a^*, c, 1}. \tag{4.8}$$

For an arbitrary $y \in S^{p^*}$, we also obtain (4.8) by applying it to $\mathcal{L}_{d_n}^*(y)$ and then passing to the limit. The second equivalence of (4.1) says that $\|y\|_{\mathcal{L}_a^*, r, 1} \approx \|y\|_{S^{p^*}}$ for any $y \in S^{p^*}$. This contradicts (4.6) and completes the proof of the first part of (4.5). The proof of the second part is similar. ■

Corollary 4.4 *Suppose that $2 < p < \infty$ (resp. $1 < p < 2$). Let $\alpha > 0$. There exists a Ritt operator T on the Schatten space S^p, with $\mathrm{Ran}(I - T)$ dense in S^p, which admits a completely bounded $H^\infty(B_\gamma)$ functional calculus with $\gamma \in \left]0, \frac{\pi}{2}\right[$ such that*

$$\sup\left\{\frac{\|x\|_{S^p}}{\|x\|_{T,c,\alpha}} \ : \ x \in S^p\right\} = \infty \quad \left(\textit{resp.}\ \sup\left\{\frac{\|x\|_{T,c,\alpha}}{\|x\|_{S^p}} \ : \ x \in S^p\right\} = \infty\right).$$

Moreover, the same result holds with $\|\cdot\|_{T,c,\alpha}$ replaced by $\|\cdot\|_{T,r,\alpha}$.

Proof : One more time, we only need to prove this result for $\alpha = 1$. Then, this follows from Lemma 4.2 and Theorem 4.3. ■

5 An alternative square function for $1 < p < 2$

Let T be a Ritt operator on $L^p(M)$, with $1 < p < 2$. For any $\alpha > 0$, we may consider an alternative square function by letting

$$\|x\|_{T,0,\alpha} = \inf\left\{\|x_1\|_{T,c,\alpha} + \|x_2\|_{T,r,\alpha} \ : \ x = x_1 + x_2\right\}$$

for any $x \in L^p(M)$.

Note that if T is both Col-Ritt and Row-Ritt, by Theorem 3.9, the square functions $\|x\|_{T,0,\alpha}$ and $\|x\|_{T,0,\beta}$ are equivalent for any $\alpha, \beta > 0$.

Suppose that $\|x\|_{T,0,\alpha}$ is finite and that we have a decomposition $x = x_1 + x_2$ with $\|x_1\|_{T,c,\alpha} < \infty$

and $\|x_2\|_{T,r,\alpha} < \infty$. Letting $u_k = k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^\alpha x_1$ and $v_k = k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^\alpha x_2$, we have

$$k^{\alpha-\frac{1}{2}}T^{k-1}(I-T)^\alpha x = u_k + v_k, \qquad k \geqslant 1.$$

Moreover, the sequences u and v belong to $L^p(M,\ell_c^2)$ and $L^p(M,\ell_r^2)$ respectively. We deduce that

$$\|x\|_{T,\alpha} \leqslant \|x\|_{T,0,\alpha}, \qquad x \in L^p(M).$$

We do not know if the two square functions $\|\cdot\|_{T,\alpha}$ and $\|\cdot\|_{T,0,\alpha}$ are equivalent in general. In the next statement, we give a sufficient condition for an such equivalence to hold true.

Theorem 5.1 *Suppose $1 < p < 2$. Let T be a bounded operator on $L^p(M)$ with $\operatorname{Ran}(I-T)$ dense in $L^p(M)$. Assume that T is both Col-Ritt and Row-Ritt. Let $\alpha, \eta > 0$. Suppose that T satisfies a 'dual square function estimate'*

$$\|y\|_{T^*,\eta} \lesssim \|y\|_{L^{p^*}(M)}, \qquad y \in L^{p^*}(M). \tag{5.1}$$

Then we have an equivalence

$$\|x\|_{T,\alpha} \approx \|x\|_{T,0,\alpha}, \qquad x \in L^p(M).$$

Indeed, there is a positive constant C such that whenever $x \in L^p(M)$ satisfies $\|x\|_{T,\alpha} < \infty$, then there exists $x_1, x_2 \in L^p(M)$ such that

$$x = x_1 + x_2 \quad \text{and} \quad \|x_1\|_{T,c,\alpha} + \|x_2\|_{T,r,\alpha} \leqslant C\|x\|_{T,\alpha}.$$

Proof : Since T is both Col-Ritt and Row-Ritt, it is also an R-Ritt operator. Then, by Theorem 3.6 and Theorem 3.9, we only need to prove this result for $\alpha = 1$ and $\eta = 1$. Observe that, for any $y \in L^{p^*}(M)$, we have

$$\begin{aligned}
&\Big\|\Big(k^{\frac{1}{2}}(T^*)^{k-1}(I+T^*)^2(I-T^*)y\Big)_{k\geqslant 1}\Big\|_{L^{p^*}(M,\ell^2_{\mathrm{rad}})}\\
&\quad\lesssim \|(I+T^*)^2\|_{L^{p^*}(M)\to L^{p^*}(M)}\Big\|\Big(k^{\frac{1}{2}}(T^*)^{k-1}(I-T^*)y\Big)_{k\geqslant 1}\Big\|_{L^{p^*}(M,\ell^2_{\mathrm{rad}})}\\
&\quad\lesssim \|y\|_{L^{p^*}(M)} \qquad \text{by (5.1).}
\end{aligned}$$

We let

$$\begin{array}{cccc}
Z: & L^{p^*}(M) & \longrightarrow & L^{p^*}(M,\ell^2_{\mathrm{rad}})\\
& y & \longmapsto & \Big(k^{\frac{1}{2}}(T^*)^{k-1}(I+T^*)^2(I-T^*)y\Big)_{k\geqslant 1}
\end{array}$$

denote the resulting bounded map. Let $x \in L^p(M)$ such that $\|x\|_{T,1} < \infty$. There exists two elements

$u \in L^p(M, \ell_c^2)$ and $v \in L^p(M, \ell_r^2)$ such that for any positive integer k

$$u_k + v_k = k^{\frac{1}{2}} T^{k-1}(I-T)x \tag{5.2}$$

and such that

$$\|u\|_{L^p(M,\ell_c^2)} + \|v\|_{L^p(M,\ell_r^2)} \leqslant 2\|x\|_{T,1}.$$

Recall that we have contractive inclusions $L^p(M, \ell_c^2) \subset L^p(M, \ell_{\mathrm{rad}}^2)$ and $L^p(M, \ell_r^2) \subset L^p(M, \ell_{\mathrm{rad}}^2)$. Thus, by (2.3), we can define x_1 and x_2 of $L^p(M)$ by

$$x_1 = Z^* u \qquad \text{and} \qquad x_2 = Z^* v.$$

We will show that $x = x_1 + x_2$. Since T is a Ritt-operator, there exists a positive constant C such that

$$\begin{aligned}\sum_{k=1}^{+\infty} \left\| k^{\frac{1}{2}} T^{k-1}(I-T)^2 \right\|^2_{L^p(M) \to L^p(M)} &= \sum_{k=1}^{+\infty} k \left\| T^{k-1}(I-T)^2 \right\|^2_{L^p(M) \to L^p(M)} \\ &\leqslant C^2 \sum_{k=1}^{+\infty} \frac{1}{k^3} < \infty.\end{aligned}$$

For any $1 < p < 2$, by [52, Proposition 2.5], we have the contractive inclusion $L^p(M, \ell_c^2) \subset \ell^2(L^p(M))$. We deduce that $\sum_{k=1}^{+\infty} \|u_k\|^2_{L^p(M)} < \infty$. According to the Cauchy-Schwarz inequality, we deduce that the series

$$\sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1} \big(I - T^2\big)^2 u_k = (I+T)^2 \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1}(I-T)^2 u_k$$

converges absolutely in $L^p(M)$. Now, for any $y \in L^{p^*}(M)$, we have

$$\begin{aligned}\Big\langle (I-T)x_1, y \Big\rangle_{L^p(M), L^{p^*}(M)} &= \Big\langle (I-T) Z^* u, y \Big\rangle_{L^p(M), L^{p^*}(M)} \\ &= \Big\langle u, Z(I-T^*) y \Big\rangle_{L^p(M, \ell^2_{\mathrm{rad}}), L^{p^*}(M, \ell^2_{\mathrm{rad}})} \\ &= \left\langle u, \Big(k^{\frac{1}{2}} (T^*)^{k-1} (I+T^*)^2 (I-T^*)^2 y \Big)_{k \geqslant 1} \right\rangle_{L^p(M, \ell^2_{\mathrm{rad}}), L^{p^*}(M, \ell^2_{\mathrm{rad}})} \\ &= \sum_{k=1}^{+\infty} \Big\langle u_k, k^{\frac{1}{2}} (T^*)^{k-1} \big(I - (T^*)^2\big)^2 y \Big\rangle_{L^p(M), L^{p^*}(M)} \\ &= \left\langle \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1} \big(I - T^2\big)^2 u_k, y \right\rangle_{L^p(M), L^{p^*}(M)}.\end{aligned}$$

Thus, we deduce that

$$(I-T)x_1 = \sum_{k=1}^{+\infty} k^{\frac{1}{2}} T^{k-1} \big(I - T^2\big)^2 u_k. \tag{5.3}$$

Similarly we have

$$(I-T)x_2=\sum_{k=1}^{+\infty}k^{\frac{1}{2}}T^{k-1}\big(I-T^2\big)^2v_k.$$

Now, we infer that

$$\begin{aligned}(I-T)(x_1+x_2)&=\sum_{k=1}^{+\infty}k^{\frac{1}{2}}T^{k-1}\big(I-T^2\big)^2u_k+\sum_{k=1}^{+\infty}k^{\frac{1}{2}}T^{k-1}\big(I-T^2\big)^2v_k\\&=\sum_{k=1}^{+\infty}k^{\frac{1}{2}}T^{k-1}\big(I-T^2\big)^2(u_k+v_k)\\&=\sum_{k=1}^{+\infty}k^{\frac{1}{2}}T^{k-1}\big(I-T^2\big)^2k^{\frac{1}{2}}T^{k-1}(I-T)x\qquad\text{by (5.2)}\\&=\sum_{k=1}^{+\infty}kT^{2k-2}(I+T)^2(I-T)^3x.\end{aligned}$$

By (4.2), for any $z\in\mathbb{D}$, we have

$$\sum_{k=1}^{+\infty}kz^{2k-2}(1-z^2)^2=1.$$

Since the operator T is power bounded, we note that for every $\varrho\in]0,1[$ we have

$$I=\sum_{k=1}^{+\infty}k(\varrho T)^{2k-2}\big(I-(\varrho T)^2\big)^2,\tag{5.4}$$

the series being absolutely convergent. Hence, for any $\varrho\in]0,1[$, we have

$$\begin{aligned}(I-\varrho T)x&=(I-\varrho T)\sum_{k=1}^{+\infty}k(\varrho T)^{2k-2}\big(I-(\varrho T)^2\big)^2x\\&=\sum_{k=1}^{+\infty}k(\varrho T)^{2k-2}(I+\varrho T)^2(I-\varrho T)^3x.\end{aligned}$$

It is not difficult to see that the latter series is normally convergent on [0,1]. Hence, letting ϱ to 1, we deduce that

$$(I-T)x=\sum_{k=1}^{+\infty}kT^{2k-2}(I+T)^2(I-T)^3x.$$

Then we obtain

$$(I-T)x=(I-T)(x_1+x_2).$$

Since the space $\operatorname{Ran}(I-T)$ is dense in $L^p(M)$, by the Mean Ergodic Theorem (see [59, Section 2.1]), the operator $I-T$ is injective. Consequently, we have $x=x_1+x_2$. Now, it remains to estimate

$\|x_1\|_{T,1,c}$ and $\|x_2\|_{T,1,r}$. According to (5.3), we have

$$m^{\frac{1}{2}}T^{m-1}(I-T)x_1 = \sum_{k=1}^{+\infty} k^{\frac{1}{2}}m^{\frac{1}{2}}T^{k+m-2}\big(I-T^2\big)^2 u_k$$

for any integer $m \geqslant 1$. It is convenient to write this as $m^{\frac{1}{2}}T^{m-1}(I-T)x_1 = (I+T)^2 y_m$ with

$$y_m = \sum_{k=1}^{+\infty} k^{\frac{1}{2}}m^{\frac{1}{2}}T^{k+m-2}(I-T)^2 u_k. \tag{5.5}$$

Now, observe that

$$k^{\frac{1}{2}}m^{\frac{1}{2}}T^{k+m-2}(I-T)^2 = \frac{k^{\frac{1}{2}}m^{\frac{1}{2}}}{(k+m-1)^2} \cdot (k+m-1)^2 T^{k+m-2}(I-T)^2.$$

According to Proposition II.2.3 and Lemma II.2.4, the matrix

$$\left[\frac{k^{\frac{1}{2}}m^{\frac{1}{2}}}{(k+m-1)^2}\right]_{k,m\geqslant 1}$$

represents an element of $B(\ell^2)$. Moreover, by Proposition 3.1, the set

$$\Big\{(k+m-1)^2 T^{k+m-2}(I-T)^2 \,:\, k,m \geqslant 1\Big\}$$

is Col-bounded. By Proposition 3.7, we deduce that $(y_m)_{m\geqslant 1} \in L^p(M, \ell^2_c)$ and that

$$\big\|(y_m)_{m\geqslant 1}\big\|_{L^p(M,\ell^2_c)} \lesssim \|u\|_{L^p(M,\ell^2_c)}.$$

Since $\{T\}$ is Col-bounded, we have

$$\begin{aligned}
\|x_1\|_{T,c,1} &= \left\|\Big(m^{\frac{1}{2}}T^{m-1}(I-T)x_1\Big)_{m\geqslant 1}\right\|_{L^p(M,\ell^2_c)} \\
&= \left\|\Big((I+T)^2 y_m\Big)_{m\geqslant 1}\right\|_{L^p(M,\ell^2_c)} \qquad \text{by (5.5)} \\
&\lesssim \big\|(y_m)_{m\geqslant 1}\big\|_{L^p(M,\ell^2_c)}.
\end{aligned}$$

Finally, we deduce that there exists a positive constant C such that

$$\|x_1\|_{T,c,1} \leqslant C\|u\|_{L^p(M,\ell^2_c)}.$$

Moreover, we have a similar result for x_2. Finally, we have

$$\begin{aligned}\|x_1\|_{T,c,1} + \|x_2\|_{T,r,1} &\leqslant C\|u\|_{L^p(M,\ell^2_c)} + C\|v\|_{L^p(M,\ell^2_r)} \\ &\leqslant C\|x\|_{T,1}.\end{aligned}$$

■

Corollary 5.2 *Suppose $1 < p < 2$. Let T be a bounded operator on $L^p(M)$ with $\mathrm{Ran}(I-T)$ dense in $L^p(M)$ and let $\alpha > 0$. Assume that T admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$. Then we have an equivalence*

$$\inf\Big\{\|x_1\|_{T,c,\alpha} + \|x_2\|_{T,r,\alpha} \ :\ x = x_1 + x_2\Big\} \approx \|x\|_{L^p(M)}, \qquad x \in L^p(M).$$

Proof : By Theorem 3.3, the operator T is both Col-Ritt and Row-Ritt (hence R-Ritt). Moreover, by Theorem 3.5, it satisfies a 'dual square estimate'

$$\|y\|_{T^*,1} \lesssim \|y\|_{L^{p^*}(M)}, \qquad y \in L^{p^*}(M).$$

Then, by Theorem 5.1 above, the norms $\|\cdot\|_{T,\alpha}$ and $\|\cdot\|_{T,0,\alpha}$ are equivalent. Furthermore, by Theorem 3.6 and (1.3), $\|\cdot\|_{T,\alpha}$ is equivalent to the usual norm $\|\cdot\|_{L^p(M)}$, which proves the result. ■

Assume now that τ is finite and normalized, that is, $\tau(1) = 1$. Following [44] and [104] (see also [4]), we say that a linear map T on M is a Markov map if T is unital, completely positive and trace preserving. As is well known, such a map is necessarily normal and for any $1 \leqslant p < \infty$, it extends to a contraction T_p on $L^p(M)$. We say that T is selfadjoint if, for any $x, x' \in M$, we have

$$\tau\big(T(x)x'\big) = \tau\big(xT(x')\big).$$

This is equivalent to T_2 being selfadjoint in the Hilbertian sense. We also consider the operator

$$A_p = I - T_p.$$

The following result is stated in [68] with *bounded* instead of *completely bounded*. But a careful reading of the proof shows that we have this stronger result. We refer to [45], [52], [67] and [68] for information on $H^\infty(\Sigma_\theta)$ functional calculus.

Proposition 5.3 *Suppose $1 < p < \infty$. Let T be a selfadjoint Markov map on M. Then the operator A_p is sectorial and admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in \left]0, \frac{\pi}{2}\right[$.*

Assume $1 < p < \infty$. At this point, it is crucial to recall that L^p-realizations T_p of Markov maps T on M such that $-1 \notin \sigma(T_2)$ are Ritt operators, as noticed by C. Le Merdy in [68]. Let T be a selfadjoint Markov map on M. According to [68] and Proposition 5.3, we obtain that T_p admits a completely bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \left]0, \frac{\pi}{2}\right[$. Hence, by Corollary 5.2, we deduce the following result which strengthens a result of [68].

Corollary 5.4 *Suppose $1 < p < 2$. Let T be a selfadjoint Markov map on M such that $-1 \notin \sigma(T_2)$ with $\mathrm{Ran}(I - T_p)$ dense in $L^p(M)$. Then, for any $\alpha > 0$ there exists a positive constant C such that for any $x \in L^p(M)$, there exists $x_1, x_2 \in L^p(M)$ satisfying $x = x_1 + x_2$ and*

$$\left\| \left(\sum_{k=1}^{+\infty} k^{2\alpha-1} \left| T^{k-1}(I-T)^{\alpha}(x_1) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} + \left\| \left(\sum_{k=1}^{+\infty} k^{2\alpha-1} \left| \left(T^{k-1}(I-T)^{\alpha}(x_2) \right)^* \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leqslant C \|x\|_{L^p(M)}.$$

Chapter IV

Noncommutative Figà-Talamanca-Herz algebras for Schur multipliers

1 Introduction

The Fourier algebra $A(G)$ of a locally compact group G was introduced by P. Eymard in [39]. The algebra $A(G)$ is the predual of the group von Neumann algebra $VN(G)$. If G is abelian with dual group $\widehat{G}$, then the fourier transform induces an isometric isomorphism of $L_1(\widehat{G})$ onto $A(G)$. In [41], A. Figà-Talamanca showed, if G is abelian, that the natural predual of the Banach space of the bounded Fourier multipliers on $L^p(G)$ is isometrically isomorphic to a space $A_p(G)$ of continuous functions on G. Moreover $A_2(G) = A(G)$ isometrically. In [39] and [47], C. Herz proved that the space $A_p(G)$ is a Banach algebra for the usual product of functions (see also [91]). Hence $A_p(G)$ is an L^p-analogue of the Fourier algebra $A(G)$. These algebras are called Figà-Talamanca-Herz algebras. In [105], V. Runde introduced an operator space analogue $OA_p(G)$ of the algebra $A_p(G)$. The underlying Banach space of $OA_p(G)$ is different from the Banach space $A_p(G)$. Moreover, it is possible to show (in using a suitable variant of [64, Theorem 5.6.1]) that $OA_p(G)$ is the natural predual of the operator space of the completely bounded Fourier multipliers. We refer to [28], [29], [63] and [106] for other operator space analogues of $A_p(G)$.

The purpose of this article is to introduce noncommutative analogues of these algebras in the context of completely bounded Schur multipliers on Schatten spaces S^p. Recall that a map $T\colon S^p \to S^p$ is completely bounded if $Id_{S^p} \otimes T$ is bounded on $S^p(S^p)$. If $1 \leqslant p < \infty$, the operator space $CB(S^p)$ of completely bounded maps from S^p into itself is naturally a dual operator space. Indeed, we have a completely isometric isomorphism $CB(S^p) = \big(S^p \widehat{\otimes} S^{p^*}\big)^*$ where $\widehat{\otimes}$ denote the operator space projective tensor product. Moreover, we will prove that the subspace $\mathfrak{M}_{p,cb}$ of completely bounded Schur multipliers is a maximal commutative subset of $CB(S^p)$. Consequently, the subspace $\mathfrak{M}_{p,cb}$ is w*-closed in $CB(S^p)$. Hence $\mathfrak{M}_{p,cb}$ is naturally a dual operator space with $\mathfrak{M}_{p,cb} = \big(S^p \widehat{\otimes} S^{p^*}/(\mathfrak{M}_{p,cb})_\perp\big)^*$. If we denote by $\psi_p\colon S^p \widehat{\otimes} S^{p^*} \to S^1$ the map $A \otimes B \mapsto A * B$, where $*$ is the Schur product, we will show that $(\mathfrak{M}_{p,cb})_\perp = \operatorname{Ker} \psi_p$. Now, we define the operator space $\mathfrak{R}_{p,cb}$ as the space $\operatorname{Im} \psi_p$ equipped with the operator space structure of $S^p \widehat{\otimes} S^{p^*}/\operatorname{Ker} \psi_p$. We have completely isometrically $\big(\mathfrak{R}_{p,cb}\big)^* = \mathfrak{M}_{p,cb}$.

Moreover, by definition, we have a completely contractive inclusion $\mathfrak{R}_{p,cb} \subset S^1$. Recall that elements of S^1 can be regarded as infinite matrices. Our principal result is the following theorem.

Theorem 1.1 *Suppose $1 \leqslant p < \infty$. The predual $\mathfrak{R}_{p,cb}$ of the operator space $\mathfrak{M}_{p,cb}$ equipped with the usual matricial product or the Schur product is a completely contractive Banach algebra.*

In [85] and [111], S. K. Parott and R. S. Strichartz showed that if $1 \leqslant p \leqslant \infty$, $p \neq 2$ every isometric Fourier multiplier on $L^p(G)$ is a scalar multiple of an operator induced by a translation. In [41], A. Figà-Talamanca showed that the space of bounded Fourier multipliers is the closure in the weak operator topology of the span of these operators. We give noncommutative analogues of these two results.

Theorem 1.2
1. *Suppose $1 \leqslant p \leqslant \infty$. If $p \neq 2$, an isometric Schur multiplier on S^p is defined by a matrix $[a_i b_j]$ with $a_i, b_j \in \mathbb{T}$.*
2. *Suppose $1 \leqslant p < \infty$. The space $\mathfrak{M}_p$ of bounded Schur multipliers on S^p is the closure of the span of isometric Schur multipliers in the weak operator topology.*

The paper is organized as follows.

In §2, we fix notations and we show that the natural preduals of $\mathfrak{M}_p$ and $\mathfrak{M}_{p,cb}$ admit concrete realizations as spaces of matrices. We give elementary properties of these spaces.

In §3, we show that the operator space $\mathfrak{R}_{p,cb}$ equipped with the matricial product is a completely contractive Banach algebra.

In §4, we turn to the Schur product. We observe that the natural predual $\mathfrak{R}_p$ of the Banach space $\mathfrak{M}_p$ of bounded Schur multipliers is a Banach algebra for the Schur product. Moreover, we show that the space $\mathfrak{R}_{p,cb}$ equipped with the Schur product is a completely contractive Banach algebra.

In §5, we determine the isometric Schur multipliers on S^p and prove that the space $\mathfrak{M}_p$ is the closure in the weak operator topology of the span of isometric multipliers.

2 Preduals of spaces of Schur multipliers

Let us recall some basic notations. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and δ_{ij} the symbol of Kronecker.

If E and F are Banach spaces, $B(E,F)$ is the space of bounded linear maps between E and F. We denote by $\otimes_\gamma$ the Banach projective tensor product. If E, F and G are Banach spaces we have $(E \otimes_\gamma F)^* = B(E, F^*)$ isometrically. In particular, if E is a dual Banach space, $B(E)$ is also a dual Banach space. If (E_0, E_1) is a compatible couple of Banach spaces we denote by $(E_0, E_1)_\theta$ the intermediate space obtained by complex interpolation between E_0 and E_1.

The readers are refereed to [13], [37], [86] and [101] for details on operator spaces and completely bounded maps. We let $CB(E,F)$ for the space of all completely bounded maps endowed with the norm

$$\|T\|_{E \longrightarrow F,cb} = \sup_{n \geqslant 1} \left\|Id_{M_n} \otimes u\right\|_{M_n(E) \longrightarrow M_n(F)}.$$

When E and F are two operator spaces, $CB(E,F)$ is an operator space for the structure corresponding to the isometric identifications $M_n(CB(E,F)) = CB(E, M_n(F))$. The dual operator space of E is $E^* = CB(E,\mathbb{C})$. If E and F are operator spaces then the adjoint map $T \mapsto T^*$ from $CB(E,F)$ into $CB(F^*,E^*)$ is a complete isometry.

If I is a set, we denote by C_I the operator space $B(\mathbb{C},\ell^2_I)$ and by R_I the operator space $B(\overline{\ell^2_I},\mathbb{C})$. We have a complete isometry $B(\ell^2_I) = CB(C_I)$ (see [13, (1.14)]).

The complex interpolated space between two compatible operator spaces E_0 and E_1 is the usual Banach space E_θ with the matrix norms corresponding to the isometric identifications $M_n(E_\theta) = \big(M_n(E_0), M_n(E_1)\big)_\theta$. Let F_0, F_1 be two other compatible operator spaces. Let $\varphi\colon E_0+E_1 \to F_0+F_1$ be a linear map. If φ is completely bounded as a map from E_0 into F_0, and from E_1 into F_1, then, for any $0 \leqslant \theta \leqslant 1$, φ is completely bounded from E_θ into F_θ with

$$\|\varphi\|_{cb,E_\theta\to F_\theta} \leqslant \big(\|\varphi\|_{cb,E_0\to F_0}\big)^{1-\theta}\big(\|\varphi\|_{cb,E_1\to F_1}\big)^{\theta}.$$

If $E_0 \cap E_1$ is dense in both E_0 and E_1, we have a completely contractive inclusion

$$\big(CB(E_0), CB(E_1)\big)_\theta \subset CB(E_\theta)$$

(see [46, Lemma 0.2]).

We denote by $\widehat{\otimes}$ the operator space projective tensor product, by $\otimes_{\min}$ the operator space minimal tensor product, by $\otimes_h$ the Haagerup tensor product, by $\otimes_{\sigma h}$ the normal Haagerup tensor product, by $\overline{\otimes}$ the normal spatial tensor product, by $\otimes_{w^*h}$ the weak* Haagerup tensor product and by $\otimes_{eh}$ the extended Haagerup tensor product (see [13], [38] and [108]). Suppose that E, F, G and H are operator spaces. If $\varphi\colon E \to F$ and $\psi\colon G \to H$ are completely bounded maps then the maps $\varphi\otimes\psi\colon E\otimes_h G \to F\otimes_h H$ and $\varphi\otimes\psi\colon E\widehat{\otimes}G \to F\widehat{\otimes}H$ are completely bounded and we have

$$\|\varphi\otimes\psi\|_{cb,E\otimes_h G\to F\otimes_h H} \leqslant \|\varphi\|_{cb,E\to F}\|\psi\|_{cb,G\to H}$$

and

$$\|\varphi\otimes\psi\|_{cb,E\widehat{\otimes}G\to F\widehat{\otimes}H} \leqslant \|\varphi\|_{cb,E\to F}\|\psi\|_{cb,G\to H}.$$

If E, F are operator spaces, we have $E\otimes_h F \subset E\otimes_{w^*h} F$ completely isometrically (see [13, page 43]).

If E, F and G are operator spaces, we denote by $CB(E\times F, G)$ the space of jointly completely bounded map. We have

$$CB(E\times F, G) = CB(E\widehat{\otimes}F, G) = CB(E, CB(F,G))$$

completely isometrically. Consequently, we have $\big(E\widehat{\otimes}F\big)^* = CB(E,F^*)$ completely isometrically. In particular, if E is a dual operator space, $CB(E)$ is also a dual operator space.

At several times, we will use the next easy lemma left to the reader.

Lemma 2.1 *Suppose E and F are operator spaces. Let $V\colon E \to F$ and $W\colon F \to E$ be any completely contractive maps. Then the map*

$$\begin{array}{rccc} \Theta_{V,W}: & CB(E) & \longrightarrow & CB(F) \\ & T & \longmapsto & VTW \end{array}$$

is completely contractive. Moreover, if E and F are reflexive then this map is also w^*-continuous.

A Banach algebra $\mathcal{A}$ equipped with an operator space structure is called completely contractive if the algebra product $(a, b) \to ab$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ is a jointly completely contractive bilinear map.

We equip ℓ^∞_I with its natural operator space structure coming from its structure as a C^*-algebra and the Banach space ℓ^1_I with its natural operator space structure coming from its structure of predual of ℓ^∞_I.

If I is an index set and if E is a vector space, we write $\mathbb{M}_I(E)$ for the space of the $I \times I$ matrices with entries in E. We denote by $\mathbb{M}^{\mathrm{fin}}_I(E)$ the subspace of matrices with a finite number of non null entries. For $I = \{1, \ldots, n\}$, we simplify the notations, we let $M_n(E)$ for $\mathbb{M}_{\{1,\ldots,n\}}(E)$. We write $\mathbb{M}_{\mathrm{fin}}$ for $\mathbb{M}^{\mathrm{fin}}_{\mathbb{N}}(\mathbb{C})$. We use the inclusion $\mathbb{M}_I \otimes \mathbb{M}_I \subset \mathbb{M}_{I\times I}$ with the identification $[A \otimes B]_{(t,r),(u,s)} = a_{tu}b_{rs}$. For all $i, j, k, l \in I$, the tensor $e_{ij} \otimes e_{kl}$ identifies to the matrix $[\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s)\in I\times I}$ (see [37, page 5] for more information on these identifications).

Given a set I, the set $\mathcal{P}_f(I)$ of all finite subsets of I is directed with respect to set inclusion. For $J \in \mathcal{P}_f(I)$ and $A \in \mathbb{M}_I$, we write $\mathcal{T}_J(A)$ for the matrix obtained from A by setting each entry to zero if its row and column index are not both in J. We call $\big(\mathcal{T}_J(A)\big)_{J\in\mathcal{P}_f(I)}$ the net of finite submatrices of A.

The Schatten-von Neumann class S^p_I, $1 \leqslant p < \infty$, is the space of those compact operators A from ℓ^2_I into ℓ^2_I such that $\|A\|_{S^p_I} = \big(\operatorname{Tr}(A^*A)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$. The space S^∞_I of compact operators from ℓ^2_I into ℓ^2_I is equipped with the operator norm. For $I = \mathbb{N}$, we simplify the notations, we let S^p for $S^p_{\mathbb{N}}$. The space $S^\infty_I(S^\infty_K)$ of compact operators from $\ell^2_I \otimes_2 \ell^2_K$ into $\ell^2_I \otimes_2 \ell^2_K$ is equipped with the operator norm. If $1 \leqslant p < \infty$, the space $S^p_I(S^p_K)$ is the space of those compact operators C from $\ell^2_I \otimes_2 \ell^2_K$ into $\ell^2_I \otimes_2 \ell^2_K$ such that $\|C\|_{S^p_I(S^p_K)} = \big((\operatorname{Tr} \otimes \operatorname{Tr})(C^*C)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$.

Elements of S^p_I are regarded as matrices $A = [a_{ij}]_{i,j\in I}$ of $\mathbb{M}_I$. If $A \in S^p_I$ we denote by A^T the operator of S^p_I whose the matrix is the matrix transpose of A. If $1 \leqslant p \leqslant \infty$, $A \in S^p_I$ and $B \in S^{p^*}_I$, the operator AB^T belongs to S^1_I. We let $\langle A, B\rangle_{S^p_I, S^{p^*}_I} = \operatorname{Tr}\big(AB^T\big)$. We have $\langle A, B\rangle_{S^p_I, S^{p^*}_I} = \lim_J \sum_{i,j\in J} a_{ij}b_{ij}$.

We equip S^∞_I with its natural operator space structure coming from its structure as a C^*-algebra. We equip S^1_I with its natural operator space structure coming from its structure as dual of S^∞_I. If $1 < p < \infty$, we give on S^p_I the operator space structure defined by $S^p_I = \big(S^\infty_I, S^1_I\big)_{\frac{1}{p}}$ completely isometrically (see [101, page 140] for interesting remarks on this definition). By the same way, we define an operator space structure on $S_I^p(S_K^p)$. We have completely isometrically $S_I^p(S_K^p) = S_K^p(S_I^p) = S_{I\times K}^p$. We will often silently use these identifications. By the same way, we define $S_I^p(S_K^p(S_L^p))$ and similar operator space structures. G. Pisier showed that a map $T\colon S_I^p \to S_I^p$ is completely bounded if $Id_{S^p} \otimes T$ is bounded on $S^p(S_I^p)$ (see [99, Lemma 1.7]). The readers are refereed to [99] for the details on operator space structures on the Schatten-von Neumann class.

We denote by $*$ the Schur (Hadamard) product: if $A = [a_{ij}]_{i,j\in I}$ and $B = [b_{ij}]_{i,j\in I}$ are matrices of $\mathbb{M}_I$ we have $A * B = [a_{ij}b_{ij}]_{i,j\in I}$. We recall that a matrix A of $\mathbb{M}_I$ defines a Schur multiplier M_A on S_I^p if for any $B \in S_I^p$ the matrix $M_A(B) = A * B$ represents an element of S_I^p. In this case, by the closed graph theorem, the linear map $B \mapsto M_A(B)$ is bounded on S_I^p. The notation $\mathfrak{M}_p^I$ stands for the algebra of all bounded Schur multipliers on the Schatten space S_I^p. We denote by $\mathfrak{M}_{p,cb}^I$ the space of completely bounded Schur multipliers on S_I^p. We give the space $\mathfrak{M}_{p,cb}^I$ the operator space structure induced by $CB(S_I^p)$. For $I = \mathbb{N}$, we simplify the notations, we let $\mathfrak{M}_p$ for $\mathfrak{M}_p^{\mathbb{N}}$ and $\mathfrak{M}_{p,cb}$ for $\mathfrak{M}_{p,cb}^{\mathbb{N}}$. Recall that if $A \in S_I^p$, we have $M_A \in \mathfrak{M}_p^I$ (see [13, page 225]).

If $M_C \in \mathfrak{M}_p^I$, we have $M_C \in \mathfrak{M}_{p^*}^I$. Moreover, if $A \in S_I^p$ and $B \in S_I^{p^*}$, we have

$$\langle M_C(A), B\rangle_{S_I^p, S_I^{p^*}} = \langle A, M_C(B)\rangle_{S_I^p, S_I^{p^*}}.$$

If $1 \leqslant p \leqslant \infty$, the Banach spaces $\mathfrak{M}_p^I$ and $\mathfrak{M}_{p^*}^I$ are isometric and the operator spaces $\mathfrak{M}_{p,cb}^I$ and $\mathfrak{M}_{p^*,cb}^I$ are completely isometric. We have $\mathfrak{M}_\infty^I = \mathfrak{M}_{\infty,cb}^I$ isometrically (see e.g. [78, Remark 2.2] and [48, Lemma 2]). Moreover, we have $\mathfrak{M}_{\infty,cb}^I = \ell_I^\infty \otimes_{w^*h} \ell_I^\infty$ completely isometrically (see e.g. [108, Theorem 3.1]) and $\mathfrak{M}_2^I = \ell_{I\times I}^\infty$ isometrically.

If $M_A \in \mathfrak{M}_p^I$ is a Schur multiplier, we have $\|M_{\mathcal{T}_J(A)}\|_{B(S_I^p)} \leqslant \|M_A\|_{B(S_I^p)}$ for any finite subset J of I. Moreover, if $M_A \in \mathfrak{M}_{p,cb}^I$, we have for any finite subset J of I the inequality $\|M_{\mathcal{T}_J(A)}\|_{CB(S_I^p)} \leqslant \|M_A\|_{CB(S_I^p)}$.

It is well-known that the map $(A, B) \mapsto A * B$ from $S_I^p \times S_I^{p^*}$ into S_I^1 is contractive. In order to study the preduals of $\mathfrak{M}_p^I$ and $\mathfrak{M}_{p,cb}^I$, we need to show that this map is jointly completely contractive.

Proposition 2.2 *Suppose* $1 \leqslant p \leqslant \infty$. *The bilinear map*

$$\begin{array}{ccc} S_I^p \times S_I^{p^*} & \longrightarrow & S_I^1 \\ (A, B) & \longmapsto & A * B \end{array}$$

is jointly completely contractive.

Proof : We denote $\beta\colon \ell_I^2 \to \ell_I^\infty$ the canonical contractive map. We have

$$\|\beta\|_{cb, C_I \to \ell_I^\infty} = \|\beta\|_{\ell_I^2 \to \ell_I^\infty} \leqslant 1 \quad \text{and} \quad \|\beta\|_{cb, R_I \to \ell_I^\infty} = \|\beta\|_{\ell_I^2 \to \ell_I^\infty} \leqslant 1$$

(see [13, (1.10)]). Then by tensoring, the map $C_I \otimes_h R_I \to \ell_I^\infty \otimes_h \ell_I^\infty$ is completely contractive. Now recall that we have a completely isometric canonical map $\ell_I^\infty \otimes_h \ell_I^\infty \to \mathfrak{M}_\infty^I$ and a completely isometric map $T \mapsto T^*$ from $CB(S_I^\infty)$ into $CB(S_I^1)$. Then the map

$$\begin{array}{ccccccc} S_I^\infty = C_I \otimes_h R_I & \longrightarrow & \ell_I^\infty \otimes_h \ell_I^\infty & \longrightarrow & \mathfrak{M}_\infty^I & \longrightarrow & CB(S_I^1) \\ e_{ij} & \longmapsto & e_i \otimes e_j & \longmapsto & M_{e_{ij}} & \longmapsto & M_{e_{ij}} \end{array}$$

is completely contractive. This means that the map $A \mapsto M_A$ from S_I^∞ into $CB(S_I^1)$ is completely contractive. Then the map $(A,B) \mapsto A*B$ from $S_I^\infty \times S_I^1$ into S_I^1 is completely jointly contractive. By the commutativity of $*$ and $\widehat{\otimes}$, the map from $S_I^1 \times S_I^\infty$ into S_I^1 is also completely jointly contractive. Finally, we obtain the result by bilinear interpolation (see [101, page 57] and [7, page 96]). ■

Then we can define the completely contractive map

$$\begin{array}{cccc} \psi_p^I : & S_I^p \widehat{\otimes} S_I^{p^*} & \longrightarrow & S_I^1 \\ & A \otimes B & \longmapsto & A * B. \end{array}$$

As $S_I^p \otimes_\gamma S_I^{p^*}$ embeds contractively into $S_I^p \widehat{\otimes} S_I^{p^*}$, the map ψ_p^I induces a contraction from $S_I^p \otimes_\gamma S_I^{p^*}$ into S_I^1, which we denote by φ_p^I. We let $\psi_p = \psi_p^{\mathbb{N}}$. The following theorem (and the comments which follow) is a noncommutative version of a theorem of Figà-Talamanca [41]. This latter theorem states that the natural predual of the space of bounded Fourier multipliers admits a concrete realization as a space $A_p(G)$ of continuous functions on G. In the sequel, we consider the dual pairs $CB(S_I^p)$, $S_I^p \widehat{\otimes} S_I^{p^*}$ and $B(S_I^p)$, $S_I^p \otimes_\gamma S_I^{p^*}$ where $1 \leqslant p < \infty$.

Theorem 2.3 *Suppose* $1 \leqslant p < \infty$.

1. *The pre-annihilator* $\big(\mathfrak{M}_{p,cb}^I\big)_\perp$ *of the space* $\mathfrak{M}_{p,cb}^I$ *of completely bounded Schur multipliers on* S_I^p *is equal to* $\operatorname{Ker} \psi_p^I$. *We have a complete isometry* $\mathfrak{M}_{p,cb}^I = \big(S_I^p \widehat{\otimes} S_I^{p^*} / \operatorname{Ker} \psi_p^I\big)^*$.
2. *The pre-annihilator* $\big(\mathfrak{M}_p^I\big)_\perp$ *of the space* $\mathfrak{M}_p^I$ *of bounded Schur multipliers on* S_I^p *is equal to* $\operatorname{Ker} \varphi_p^I$. *We have an isometry* $\mathfrak{M}_p^I = \big(S_I^p \otimes_\gamma S_I^{p^*} / \operatorname{Ker} \varphi_p^I\big)^*$.

Proof : We will only prove the part 1. The proof of part 2 is similar. Let $C = \sum_{k=1}^l A_k \otimes B_k \in S_I^p \otimes S_I^{p^*}$. Note that, for all integers k, we have $M_{A_k} \in \mathfrak{M}_p^I$. If i,j are elements of I we have

$$\begin{aligned} \langle M_{e_{ij}}, C \rangle_{CB(S_I^p), S_I^p \widehat{\otimes} S_I^{p^*}} &= \left\langle M_{e_{ij}}, \sum_{k=1}^l A_k \otimes B_k \right\rangle_{CB(S_I^p), S_I^p \widehat{\otimes} S_I^{p^*}} \\ &= \sum_{k=1}^l \langle e_{ij} * A_k, B_k \rangle_{S_I^p, S_I^{p^*}} \\ &= \sum_{k=1}^l \langle e_{ij}, A_k * B_k \rangle_{S_I^p, S_I^{p^*}} \\ &= \left\langle e_{ij}, \sum_{k=1}^l A_k * B_k \right\rangle \\ &= \Big[\psi_p^I(C)\Big]_{ij}. \end{aligned}$$

By continuity, if $C \in S^p_I \widehat{\otimes} S^{p^*}_I$, we have $\langle M_{e_{ij}}, C\rangle_{CB(S^p_I), S^p_I \widehat{\otimes} S^{p^*}_I} = \big[\psi^I_p(C)\big]_{ij}$. We deduce that, if $C \in \operatorname{Ker} \psi^I_p$ and $M_D \in \mathfrak{M}^I_{p,cb}$, we have for all $J \in \mathcal{P}_f(I)$

$$\langle M_{\mathcal{T}_J(D)}, C\rangle_{CB(S^p_I), S^p_I \widehat{\otimes} S^{p^*}_I} = 0.$$

Now, it is easy to see that we have $M_{\mathcal{T}_J(D)} \xrightarrow[J]{so} M_D$ in $CB(S^p_I)$ (i.e., for all $A \in S^p_I$, we have $M_{\mathcal{T}_J(D)}(A) \xrightarrow[J]{} M_D(A)$). Then $M_{\mathcal{T}_J(D)} \xrightarrow[J]{wo} M_D$ in $CB(S^p_I)$. Moreover, recall that, for all $J \in \mathcal{P}_f(I)$, we have $\|M_{\mathcal{T}_J(D)}\|_{\mathfrak{M}^I_{p,cb}} \leqslant \|M_D\|_{\mathfrak{M}^I_{p,cb}}$. Thus $M_{\mathcal{T}_J(D)} \xrightarrow[J]{w^*} M_D$. Consequently, if $C \in \operatorname{Ker} \psi^I_p$ and $M_D \in \mathfrak{M}^I_{p,cb}$ we have

$$\langle M_D, C\rangle_{CB(S^p_I), S^p_I \widehat{\otimes} S^{p^*}_I} = \lim_J \langle M_{\mathcal{T}_J(D)}, C\rangle_{CB(S^p_I), S^p_I \widehat{\otimes} S^{p^*}_I} = 0.$$

Thus we have $\operatorname{Ker} \psi^I_p \subset \big(\mathfrak{M}^I_{p,cb}\big)_\perp$. Now we will show that $\big(\operatorname{Ker} \psi^I_p\big)^\perp \subset \mathfrak{M}^I_{p,cb}$. Suppose that $T \in \big(\operatorname{Ker} \psi^I_p\big)^\perp$. If i, j, k, l are elements of I such that $(i, j) \neq (k, l)$, the tensor $e_{ij} \otimes e_{kl}$ belongs to $\operatorname{Ker} \psi^I_p$. Therefore we have

$$\begin{aligned} \langle T(e_{ij}), e_{kl}\rangle_{S^p_I, S^{p^*}_I} &= \langle T, e_{ij} \otimes e_{kl}\rangle_{CB(S^p_I), S^p_I \widehat{\otimes} S^{p^*}_I} \\ &= 0. \end{aligned}$$

Hence T is a Schur multiplier. We conclude that $\big(\operatorname{Ker} \psi^I_p\big)^\perp \subset \mathfrak{M}^I_{p,cb}$. Since $\operatorname{Ker} \psi^I_p$ is norm-closed in $S^p_I \widehat{\otimes} S^{p^*}_I$ we deduce that

$$\big(\mathfrak{M}^I_{p,cb}\big)_\perp \subset \Big(\big(\operatorname{Ker} \psi^I_p\big)^\perp\Big)_\perp = \operatorname{Ker} \psi^I_p.$$

Then the first claim of part 1 of the theorem is proved.

Now, we will show that $\mathfrak{M}^I_{p,cb}$ is a maximal commutative subset of $CB(S^p_I)$. Let $T\colon S^p_I \to S^p_I$ be a bounded map which commutes with all Schur multipliers $M_{e_{ij}}\colon S^p_I \to S^p_I$ where $i, j \in I$. Then, for all $i, j, k, l \in I$ such that $(i, j) \neq (k, l)$ we have

$$\begin{aligned} \langle T(e_{ij}), e_{kl}\rangle_{S^p_I, S^{p^*}_I} &= \langle T M_{e_{ij}}(e_{ij}), e_{kl}\rangle_{S^p_I, S^{p^*}_I} \\ &= \langle M_{e_{ij}} T(e_{ij}), e_{kl}\rangle_{S^p_I, S^{p^*}_I} \\ &= \langle T(e_{ij}), M_{e_{ij}}(e_{kl})\rangle_{S^p_I, S^{p^*}_I} \\ &= 0. \end{aligned}$$

Hence T is a Schur multiplier. This proves the claim. Then $\mathfrak{M}^I_{p,cb}$ is weak* closed in $CB(S^p_I)$. We immediately deduce the second claim of part 1 of the theorem. ■

If $1 \leqslant p < \infty$, we define the operator space $\mathfrak{R}^I_{p,cb}$ as the space $\operatorname{Im} \psi^I_p$ equipped with the operator space structure of $S^p_I \widehat{\otimes} S^{p^*}_I / \operatorname{Ker} \psi^I_p$. We let $\mathfrak{R}_{p,cb} = \mathfrak{R}^{\mathbb{N}}_{p,cb}$. We have completely isometrically $\big(\mathfrak{R}^I_{p,cb}\big)^* =$

$\mathfrak{M}^I_{p,cb}$. By definition, we have a completely contractive inclusion $\mathfrak{R}^I_{p,cb} \subset S^1_I$. We define the Banach space $\mathfrak{R}^I_p$ as the space $\operatorname{Im}\varphi^I_p$ equipped with the norm of $S^p_I \otimes_\gamma S^{p^*}_I/\operatorname{Ker}\varphi^I_p$. We let $\mathfrak{R}_p = \mathfrak{R}^{\mathbb{N}}_p$. We have isometrically $\big(\mathfrak{R}^I_p\big)^* = \mathfrak{M}^I_p$.

By duality, well-known results on $\mathfrak{M}^I_p$ and $\mathfrak{M}^I_{p,cb}$ translate immediately into results on $\mathfrak{R}^I_p$ and $\mathfrak{R}^I_{p,cb}$. If $1 \leqslant p < \infty$, there is a contractive inclusion $\mathfrak{R}^I_p \subset \mathfrak{R}^I_{p,cb}$. If $1 < p < \infty$, the Banach spaces $\mathfrak{R}^I_p$ and $\mathfrak{R}^I_{p^*}$ are isometric and the operator spaces $\mathfrak{R}^I_{p,cb}$ and $\mathfrak{R}^I_{p^*,cb}$ are completely isometric. We have a completely isometric isomorphism

$$\begin{array}{ccc} \ell^1_I \otimes_h \ell^1_I & \longrightarrow & \mathfrak{R}^I_{1,cb} \\ e_i \otimes e_j & \longmapsto & e_{ij} \end{array} \tag{2.1}$$

and isometric isomorphisms

$$\begin{array}{ccc} \ell^1_I \otimes_h \ell^1_I & \longrightarrow & \mathfrak{R}^I_1 \\ e_i \otimes e_j & \longmapsto & e_{ij} \end{array} \quad \text{and} \quad \begin{array}{ccc} \ell^{I\times I}_1 & \longrightarrow & \mathfrak{R}^I_2 = \mathfrak{R}^I_{2,cb} \\ e_{ij} & \longmapsto & e_{ij}. \end{array}$$

Suppose $1 \leqslant p \leqslant q \leqslant 2$, we have injective contractive maps

$$\mathfrak{M}^I_1 \subset \mathfrak{M}^I_p \subset \mathfrak{M}^I_q \subset \mathfrak{M}^I_2 \quad \text{and} \quad \mathfrak{M}^I_{1,cb} \subset \mathfrak{M}^I_{p,cb} \subset \mathfrak{M}^I_{q,cb} \subset \mathfrak{M}^I_{2,cb}$$

(see [46, page 219]). One more time, by duality, we deduce that we have injective contractive inclusions

$$\mathfrak{R}^I_2 \subset \mathfrak{R}^I_q \subset \mathfrak{R}^I_p \subset \mathfrak{R}^I_1 \quad \text{and} \quad \mathfrak{R}^I_{2,cb} \subset \mathfrak{R}^I_{q,cb} \subset \mathfrak{R}^I_{p,cb} \subset \mathfrak{R}^I_{1,cb}.$$

Actually, the last inclusions are completely contractive. It is a part of Proposition 2.7.

Suppose $1 \leqslant p < \infty$. By a well-known property of the Banach projective tensor product, an element C in S^1_I belongs to $\mathfrak{R}^I_p$ if and only if there exists two sequences $(A_n)_{n\geqslant 1} \subset S^p_I$ and $(B_n)_{n\geqslant 1} \subset S^{p^*}_I$ such that the series $\sum_{n=1}^{+\infty} A_n \otimes B_n$ converges absolutely in $S^p_I \widehat{\otimes} S^{p^*}_I$ and $C = \sum_{n=1}^{+\infty} A_n * B_n$ in S^1_I. Moreover, we have

$$\|C\|_{\mathfrak{R}^I_p} = \inf\left\{ \sum_{n=1}^{+\infty} \|A_n\|_{S^p_I} \|B_n\|_{S^{p^*}_I} \mid C = \sum_{n=1}^{+\infty} A_n * B_n \right\} \tag{2.2}$$

where the infimum is taken over all possible ways to represent C as before. We observe that we have an inclusion $\mathbb{M}^{\mathrm{fin}}_I \subset \mathfrak{R}^I_p$. It is clear that $\mathbb{M}^{\mathrm{fin}}_I$ is dense in $\mathfrak{R}^I_p$ and $\mathfrak{R}^I_{p,cb}$.

Remark 2.4 *The Banach spaces $\mathfrak{M}^I_p$ and $\mathfrak{M}^I_{p,cb}$ contain the space ℓ^∞_I. We deduce that, if I is infinite, the Banach spaces $\mathfrak{M}^I_p$, $\mathfrak{M}^I_{p,cb}$, $\mathfrak{R}^I_p$ and $\mathfrak{R}^I_{p,cb}$ are not reflexive.*

Now we make precise the duality between the operator spaces $\mathfrak{M}^I_{p,cb}$ and $\mathfrak{R}^I_{p,cb}$ on the one hand and the Banach spaces $\mathfrak{M}^I_p$ and $\mathfrak{R}^I_p$ on the other hand. Moreover, the next lemma specifies the density of $\mathbb{M}^{\mathrm{fin}}_I$ in $\mathfrak{R}^I_p$ and $\mathfrak{R}^I_{p,cb}$.

Lemma 2.5 *Suppose* $1 \leqslant p < \infty$.

1. *If J is a finite subset of I, the truncation map $\mathcal{T}_J \colon \mathfrak{R}^I_{p,cb} \to \mathfrak{R}^I_{p,cb}$ is completely contractive. Moreover, if $A \in \mathfrak{R}^I_{p,cb}$, we have in $\mathfrak{R}^I_{p,cb}$*
$$\mathcal{T}_J(A) \underset{J}{\to} A. \tag{2.3}$$

2. *For any completely bounded Schur multiplier $M_A \in \mathfrak{M}^I_{p,cb}$ and any $B \in \mathfrak{R}^I_{p,cb}$, we have*
$$\langle M_A, B\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}} = \lim_J \sum_{i,j\in J} a_{ij}b_{ij}. \tag{2.4}$$

3. *If J is a finite subset of I, the truncation map $\mathcal{T}_J \colon \mathfrak{R}^I_p \to \mathfrak{R}^I_p$ is contractive. Moreover, if $A \in \mathfrak{R}^I_p$, we have $\mathcal{T}_J(A) \underset{J}{\to} A$ in $\mathfrak{R}^I_p$.*

4. *For any bounded Schur multiplier $M_A \in \mathfrak{M}^I_p$ and any $B \in \mathfrak{R}^I_p$, we have $\langle M_A, B\rangle_{\mathfrak{M}^I_p,\mathfrak{R}^I_p} = \lim\limits_J \sum\limits_{i,j\in J} a_{ij}b_{ij}$.*

Proof : We only prove the assertions for the operator space $\mathfrak{R}^I_{p,cb}$. If i,j are elements of I and $M_A \in \mathfrak{M}^I_{p,cb}$, we have
$$\begin{aligned}\langle M_A, e_{ij}\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}} &= \langle M_A, e_{ij} * e_{ij}\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}}\\ &= \langle M_A(e_{ij}), e_{ij}\rangle_{S^p_I,S^{p^*}_I}\\ &= a_{ij}.\end{aligned}$$

Then we deduce that, for all $M_A \in \mathfrak{M}^I_{p,cb}$ and all $B \in \mathbb{M}^{\mathrm{fin}}_I$, we have $\langle M_A, B\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}} = \sum_{i,j\in I} a_{ij}b_{ij}$. Now, it is not difficult to see that, for any finite subset J of I, the truncation map $\mathcal{T}_J : S^p_I \to S^p_I$ is completely contractive. Then, it follows easily that the truncation map $\mathcal{T}_J \colon \mathfrak{M}^I_{p,cb} \to \mathfrak{M}^I_{p,cb}$ is completely contractive. Hence, by duality and by using the density of $\mathbb{M}^{\mathrm{fin}}_I$ in $\mathfrak{R}^I_{p,cb}$, we deduce that the truncation map $\mathcal{T}_J \colon \mathfrak{R}^I_{p,cb} \to \mathfrak{R}^I_{p,cb}$ is completely contractive. Furthermore, by density of $\mathbb{M}^{\mathrm{fin}}_I$ in $\mathfrak{R}^I_{p,cb}$, it is not difficult to prove the assertion (2.3). Finally, the equality (2.4) is now immediate. ■

Finally, we end the section by giving supplementary properties of these operator spaces. For that, we need the following proposition inspired by [78, Proposition 2.4]. If $x,y \in \mathbb{R}$, we denote by $M_{x,y} \colon S^p_I \to S^p_I$ the Schur multiplier associated with the matrix $\big[e^{ixr}e^{iys}\big]_{r,s\in I}$ of $\mathbb{M}_I$ and by $\overline{M}_{x,y} \colon S^p_I \to S^p_I$ the Schur multiplier associated with the matrix $\big[e^{-ixr}e^{-iys}\big]_{r,s\in I}$ of $\mathbb{M}_I$. It is easy to see that, for all $x,y \in \mathbb{R}$, the maps $M_{x,y} \colon S^p_I \to S^p_I$ and $\overline{M}_{x,y} \colon S^p_I \to S^p_I$ are completely contractive. We denote by dx the normalized measure on $[0,2\pi]$.

Proposition 2.6 *Suppose* $1 \leqslant p \leqslant \infty$. *The space $\mathfrak{M}^I_{p,cb}$ of completely bounded Schur multipliers on S^p_I is 1-completely complemented in the space $CB(S^p_I)$.*

Proof : Let $T\colon S^p_I \to S^p_I$ be a completely bounded map. For any $A \in \mathbb{M}^{\mathrm{fin}}_I$ the map

$$\begin{array}{ccc} [0,2\pi]\times[0,2\pi] & \longrightarrow & S^p_I \\ (x,y) & \longmapsto & M_{x,y}T\overline{M}_{x,y}(A) \end{array}$$

is continuous and we have

$$\begin{aligned}\left\| \int_0^{2\pi}\int_0^{2\pi} M_{x,y}T\overline{M}_{x,y}(A)dxdy \right\|_{S^p_I} &\leqslant \int_0^{2\pi}\int_0^{2\pi} \left\| M_{x,y}T\overline{M}_{x,y}(A)\right\|_{S^p_I} dxdy \\ &\leqslant \int_0^{2\pi}\int_0^{2\pi} \|M_{x,y}\|_{S^p_I\to S^p_I}\|T\|_{S^p_I\to S^p_I}\|\overline{M}_{x,y}\|_{S^p_I\to S^p_I}\|A\|_{S^p_I}dxdy \\ &\leqslant \|T\|_{S^p_I\to S^p_I}\|A\|_{S^p_I}.\end{aligned}$$

By the previous computation, we deduce that there exists a unique linear map $P(T)\colon S^p_I \to S^p_I$ such that for all $A \in S^p_I$ we have

$$\big(P(T)\big)(A) = \int_0^{2\pi}\int_0^{2\pi} M_{x,y}T\overline{M}_{x,y}(A)dxdy.$$

Moreover, for all $\sum_{k=1}^l A_k \otimes B_k \in \mathbb{M}_{\mathrm{fin}} \otimes S^p_I$ we have

$$\begin{aligned}\left\|\Big(Id_{S_p}\otimes P(T)\Big)\left(\sum_{k=1}^l A_k\otimes B_k\right)\right\|_{S^p(S^p_I)} &= \left\|\sum_{k=1}^l A_k \otimes \int_0^{2\pi}\int_0^{2\pi} M_{x,y}T\overline{M}_{x,y}(B_k)dxdy\right\|_{S^p(S^p_I)} \\ &= \left\|\int_0^{2\pi}\int_0^{2\pi}\Big(Id_{S_p}\otimes M_{x,y}T\overline{M}_{x,y}\Big)\left(\sum_{k=1}^l A_k\otimes B_k\right)dxdy\right\|_{S^p(S^p_I)} \\ &\leqslant \|T\|_{cb,S^p_I\to S^p_I}\left\|\sum_{k=1}^l A_k\otimes B_k\right\|_{S^p(S^p_I)}.\end{aligned}$$

Thus we see that the map $P(T)$ is actually completely bounded and that we have $\|P(T)\|_{cb,S^p_I\to S^p_I} \leqslant \|T\|_{cb,S^p_I\to S^p_I}$. Now, for all $r,s,k,l \in I$ we have

$$\begin{aligned}\big\langle P(T)e_{rs}, e_{kl}\big\rangle_{S^p_I,S^{p^*}_I} &= \int_0^{2\pi}\int_0^{2\pi}\Big\langle M_{x,y}T\overline{M}_{x,y}e_{rs}, e_{kl}\Big\rangle_{S^p_I,S^{p^*}_I}dxdy \\ &= \int_0^{2\pi}\int_0^{2\pi} e^{-ixr}e^{-iys}\Big\langle M_{x,y}Te_{rs}, e_{kl}\Big\rangle_{S^p_I,S^{p^*}_I}dxdy \\ &= \left(\int_0^{2\pi}\int_0^{2\pi} e^{-ixr}e^{-iys}e^{ixk}e^{iyl}dxdy\right)\big\langle Te_{rs}, e_{kl}\big\rangle_{S^p_I,S^{p^*}_I} \\ &= \left(\int_0^{2\pi} e^{ix(k-r)}dx\right)\left(\int_0^{2\pi} e^{iy(l-s)}dy\right)\big\langle Te_{rs}, e_{kl}\big\rangle_{S^p_I,S^{p^*}_I}\end{aligned}$$

$$= \delta_{rk}\delta_{sl}\big\langle T(e_{rs}), e_{kl}\big\rangle_{S_I^p, S_I^{p^*}}.$$

Then the linear map $P(T)\colon S_I^p \to S_I^p$ is a Schur multiplier. Moreover, if $T\colon S_I^p \to S_I^p$ is a Schur multiplier, we have $P(T) = T$.

Now, if $T \in M_n(CB(S_I^p))$ and $[A_{kl}]_{1\leqslant k,l\leqslant m} \in M_m(S_I^p)$, with the notations of Lemma 2.1, we have

$$\begin{aligned}
&\bigg\|\Big[\int_0^{2\pi}\int_0^{2\pi} M_{x,y}T_{ij}\overline{M}_{x,y}(A_{kl})dxdy\Big]_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant k,l\leqslant m}}\bigg\|_{M_{mn}(S_I^p)}\\
&\qquad\leqslant \int_0^{2\pi}\int_0^{2\pi}\Big\|\big[M_{x,y}T_{ij}\overline{M}_{x,y}\big]_{1\leqslant i,j\leqslant n}\Big\|_{M_n(CB(S_I^p))}\big\|[A_{kl}]\big\|dxdy\\
&\qquad= \int_0^{2\pi}\int_0^{2\pi}\Big\|\Big(Id_{M_n}\otimes\Theta_{M_{x,y},\overline{M}_{x,y}}\Big)(T)\Big\|_{M_n(CB(S_I^p))}\big\|[A_{kl}]\big\|dxdy\\
&\qquad\leqslant \|T\|_{M_n(CB(S_I^p))}\big\|[A_{kl}]_{1\leqslant k,l\leqslant m}\big\|_{M_m(S_I^p)} \qquad\text{by Lemma 2.1.}
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\big\|(Id_{M_n}\otimes P)(T)\big\|_{M_n(CB(S_I^p))} &= \Big\|\big[P(T_{ij})\big]_{1\leqslant i,j\leqslant n}\Big\|_{M_n(CB(S_I^p))}\\
&\leqslant \|T\|_{M_n(CB(S_I^p))}.
\end{aligned}$$

We deduce that the map $P\colon CB(S_I^p) \to \mathfrak{M}_{p,cb}^I$ is completely contractive. The proof is complete. ∎

Proposition 2.7 *1. We have completely isometric isomorphisms*

$$\begin{array}{ccc} \ell_I^1\widehat{\otimes}\ell_I^1 & \longrightarrow & \mathfrak{R}_{2,cb}^I \\ e_i\otimes e_j & \longmapsto & e_{ij}\end{array} \quad\textit{and}\quad \begin{array}{ccc} \ell_{I\times I}^\infty & \longrightarrow & \mathfrak{M}_{2,cb}^I \\ A & \longmapsto & M_A.\end{array}$$

2. Suppose $1 \leqslant p \leqslant q \leqslant 2$. We have injective completely contractive maps

$$\mathfrak{M}_{1,cb}^I \subset \mathfrak{M}_{p,cb}^I \subset \mathfrak{M}_{q,cb}^I \subset \mathfrak{M}_{2,cb}^I \quad\textit{and}\quad \mathfrak{R}_{2,cb}^I \subset \mathfrak{R}_{q,cb}^I \subset \mathfrak{R}_{p,cb}^I \subset \mathfrak{R}_{1,cb}^I.$$

Proof : 1) By minimality, we have a completely contractive map $\mathfrak{M}_{2,cb}^I \to \ell_{I\times I}^\infty$. We will show that the inverse map is completely contractive. We have a complete isometry

$$\begin{array}{ccc} \ell_{I\times I}^\infty & \longrightarrow & B(S_I^2) = CB(C_{I\times I}) \\ A & \longmapsto & M_A.\end{array}$$

Now we know that $(R_{I\times I})^* = C_{I\times I}$. Then we deduce a complete isometry

$$\begin{array}{ccccc} \ell_{I\times I}^\infty & \longrightarrow & CB(C_{I\times I}) & \longrightarrow & CB(R_{I\times I}) \\ A & \longmapsto & M_A & \longmapsto & (M_A)^* = M_A.\end{array}$$

By interpolation, we deduce a complete contraction

$$\ell^\infty_{I\times I} \to \big(CB(C_{I\times I}), CB(R_{I\times I})\big)_{\frac{1}{2}}.$$

Recall that we have $(C_{I\times I}, R_{I\times I})_{\frac{1}{2}} = S^2_I$ completely isometrically (see [99] pages 137 and 140). Then we have a complete contraction

$$\big(CB(C_{I\times I}), CB(R_{I\times I})\big)_{\frac{1}{2}} \to CB(S^2_I).$$

Finally, we obtain a complete contraction $\ell^\infty_{I\times I} \to CB(S^2_I)$. We obtain the other isomorphism by duality.

2) Let $1 \leqslant p \leqslant q \leqslant 2$. Recall that we have a contraction from $\mathfrak{M}^I_{p,cb}$ into $\mathfrak{M}^I_{2,cb}$ (see [46, page 219]). Moreover we have $\mathfrak{M}^I_{2,cb} = \ell^\infty_{I\times I}$ completely isometrically. Thus we have a complete contraction $\mathfrak{M}^I_{p,cb} \to \mathfrak{M}^I_{2,cb}$. Now, there exists $0 \leqslant \theta \leqslant 1$ with $S^q_I = (S^p_I, S^2_I)_\theta$. Moreover, the identity mapping $\mathfrak{M}^I_{p,cb} \to \mathfrak{M}^I_{p,cb}$ is completely contractive. By interpolation, we obtain a complete contraction $\mathfrak{M}^I_{p,cb} \to (\mathfrak{M}^I_{p,cb}, \mathfrak{M}^I_{2,cb})_\theta$. On one hand, we know that we have a complete contraction

$$\Big(CB(S^p_I), CB(S^2_I)\Big)_\theta \to CB\Big((S^p_I, S^2_I)_\theta\Big) = CB(S^q_I).$$

On the other hand, the space $\mathfrak{M}^I_{p,cb}$ of completely bounded Schur multipliers is 1-completely complemented in the space $CB(S^p_I)$. Then we have a complete contraction $(\mathfrak{M}^I_{p,cb}, \mathfrak{M}^I_{2,cb})_\theta \to \mathfrak{M}^I_{q,cb}$. By composition, we deduce that we have a complete contraction $\mathfrak{M}^I_{p,cb} \subset \mathfrak{M}^I_{q,cb}$. We obtain the other completely contractive maps by duality. ■

3 Noncommutative Figà-Talamanca-Herz algebras

We begin with the cases $p = 1$ and $p = 2$. Recall that we have a completely isometric isomorphism $\mathfrak{R}^I_{1,cb} = \ell^1_I \otimes_h \ell^1_I$ (see (2.1)) and a completely contractive inclusion $\mathfrak{R}^I_{1,cb} \subset S^1_I$. Hence, the trace on S^1_I induces a completely contractive functional

$$\begin{array}{rccc} \mathrm{Tr}: & \ell^1_I \otimes_h \ell^1_I & \longrightarrow & \mathbb{C} \\ & e_i \otimes e_j & \longmapsto & \delta_{ij}. \end{array}$$

By tensoring, we deduce a completely contractive map

$$Id_{\ell^1_I} \otimes \mathrm{Tr} \otimes Id_{\ell^1_I} : \ell^1_I \otimes_h \ell^1_I \otimes_h \ell^1_I \otimes_h \ell^1_I \to \ell^1_I \otimes_h \ell^1_I.$$

By composition with the canonical completely contractive map

$$(\ell_I^1 \otimes_h \ell_I^1)\widehat{\otimes}(\ell_I^1 \otimes_h \ell_I^1) \to \ell_I^1 \otimes_h \ell_I^1 \otimes_h \ell_I^1 \otimes_h \ell_I^1$$

we obtain a completely contractive map

$$Id_{\ell_I^1} \otimes \mathrm{Tr} \otimes Id_{\ell_I^1} \colon (\ell_I^1 \otimes_h \ell_I^1)\widehat{\otimes}(\ell_I^1 \otimes_h \ell_I^1) \to \ell_I^1 \otimes_h \ell_I^1.$$

With the identification $\mathfrak{R}_{1,cb}^I = \ell_I^1 \otimes_h \ell_I^1$, we obtain the completely contractive map

$$\begin{array}{ccc} \mathfrak{R}_{1,cb}^I \widehat{\otimes} \mathfrak{R}_{1,cb}^I & \longrightarrow & \mathfrak{R}_{1,cb}^I \\ A \otimes B & \longmapsto & AB. \end{array}$$

This means that the space $\mathfrak{R}_{1,cb}^I$ equipped with the matricial product is a completely contractive Banach algebra. Now, recall that we have $\mathfrak{R}_{2,cb}^I = \ell_I^1 \widehat{\otimes} \ell_I^1$ completely isometrically. Then, by a similar argument, $\mathfrak{R}_{2,cb}^I$ equipped with the matricial product is also a completely contractive Banach algebra. For other values of p, the proof is more complicated since we do not have any explicit description of $\mathfrak{R}_{p,cb}^I$.

In the following proposition, we give a link between $\mathfrak{R}_{p,cb}^I$ and $\mathfrak{R}_{p,cb}^{I\times I}$.

Proposition 3.1 *Suppose $1 \leqslant p < \infty$. Then there exists a canonical complete contraction*

$$\begin{array}{ccc} \mathfrak{R}_{p,cb}^I \widehat{\otimes} \mathfrak{R}_{p,cb}^I & \longrightarrow & \mathfrak{R}_{p,cb}^{I\times I} \\ A \otimes B & \longmapsto & A \otimes B. \end{array}$$

Proof : The identity mapping on $S_I^p \otimes S_I^p$ extends to a complete contraction $S_I^p \widehat{\otimes} S_I^p \to S_I^p(S_I^p)$. Hence by tensoring, we obtain a completely contractive map

$$\beta \colon S_I^p \widehat{\otimes} S_I^p \widehat{\otimes} S_I^{p^*} \widehat{\otimes} S_I^{p^*} \to S_I^p(S_I^p) \widehat{\otimes} S_I^{p^*}(S_I^{p^*}).$$

The map $\psi_p^I \colon S_I^p \widehat{\otimes} S_I^{p^*} \to \mathfrak{R}_{p,cb}^I$ is a complete quotient map. By [37, Proposition 7.1.7], we obtain a complete quotient map

$$\psi_p^I \otimes \psi_p^I \colon S_I^p \widehat{\otimes} S_I^{p^*} \widehat{\otimes} S_I^p \widehat{\otimes} S_I^{p^*} \to \mathfrak{R}_{p,cb}^I \widehat{\otimes} \mathfrak{R}_{p,cb}^I.$$

Finally, by the commutativity of $\widehat{\otimes}$, the map

$$\begin{array}{cccc} \alpha \colon & S_I^p \widehat{\otimes} S_I^{p^*} \widehat{\otimes} S_I^p \widehat{\otimes} S_I^{p^*} & \longrightarrow & S_I^p \widehat{\otimes} S_I^p \widehat{\otimes} S_I^{p^*} \widehat{\otimes} S_I^{p^*} \\ & A \otimes B \otimes C \otimes D & \longmapsto & A \otimes C \otimes B \otimes D \end{array}$$

is completely isometric. We will prove that there exists a unique linear map such that the following diagram is commutative and that this map is completely contractive.

$$\begin{tikzcd}
S^p_I \widehat{\otimes} S^{p^*}_I \widehat{\otimes} S^p_I \widehat{\otimes} S^{p^*}_I \arrow[r, "\alpha"] \arrow[d, "\psi^I_p \otimes \psi^I_p"'] & S^p_I \widehat{\otimes} S^p_I \widehat{\otimes} S^{p^*}_I \widehat{\otimes} S^{p^*}_I \arrow[r, "\beta"] & S^p_I(S^p_I) \widehat{\otimes} S^{p^*}_I(S^{p^*}_I) \arrow[d, "\psi^{I \times I}_p"] \\
\mathfrak{R}^I_{p,cb} \widehat{\otimes} \mathfrak{R}^I_{p,cb} \arrow[rr, dotted] & & \mathfrak{R}^{I \times I}_{p,cb}
\end{tikzcd}$$

We have $\mathfrak{R}^I_{p,cb} \widehat{\otimes} \mathfrak{R}^I_{p,cb} = \big(S^p_I \widehat{\otimes} S^{p^*}_I \widehat{\otimes} S^p_I \widehat{\otimes} S^{p^*}_I\big) / \mathrm{Ker}\,(\psi^I_p \otimes \psi^I_p)$ completely isometrically. It suffices to show that $\mathrm{Ker}\,(\psi^I_p \otimes \psi^I_p) \subset \mathrm{Ker}\,(\psi^{I \times I}_p \beta\alpha)$. By [37, Proposition 7.1.7] , we have the equality

$$\mathrm{Ker}\,(\psi^I_p \otimes \psi^I_p) = \mathrm{closure}\,\Big(\mathrm{Ker}\,(\psi^I_p) \otimes S^p_I \widehat{\otimes} S^{p^*}_I + S^p_I \widehat{\otimes} S^{p^*}_I \otimes \mathrm{Ker}\,(\psi^I_p)\Big).$$

Since the space $\mathrm{Ker}\,(\psi^{I \times I}_p \beta\alpha)$ is closed in $S^p_I \widehat{\otimes} S^{p^*}_I \widehat{\otimes} S^p_I \widehat{\otimes} S^{p^*}_I$, it suffices to show that

$$\mathrm{Ker}\,(\psi^I_p) \otimes S^p_I \widehat{\otimes} S^{p^*}_I + S^p_I \widehat{\otimes} S^{p^*} \otimes \mathrm{Ker}\,(\psi^I_p) \subset \mathrm{Ker}\,(\psi^{I \times I}_p \beta\alpha).$$

Let $E \in \mathrm{Ker}\,(\psi^I_p) \otimes S^p_I \widehat{\otimes} S^{p^*}_I$. There exists integers n_i, m_j, matrices $A_{k,i}, C_{l,j} \in S^p_I$ and $B_{k,i}, D_{l,j} \in S^{p^*}_I$ such that the sequences

$$\left(\sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i}\right)_{i \geqslant 1} \quad \text{and} \quad \left(\sum_{l=1}^{m_j} C_{l,j} \otimes D_{l,j}\right)_{j \geqslant 1}$$

are convergent in $S^p_I \widehat{\otimes} S^{p^*}_I$,

$$E = \left(\lim_{i \to +\infty} \sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i}\right) \otimes \left(\lim_{j \to +\infty} \sum_{l=1}^{m_j} C_{l,j} \otimes D_{l,j}\right)$$

and

$$\psi^I_p\left(\lim_{i \to +\infty} \sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i}\right) = 0.$$

Then, in the space S^1_I, we have

$$\sum_{k=1}^{n_i} A_{k,i} * B_{k,i} \xrightarrow[i \to +\infty]{} 0. \tag{3.1}$$

Moreover, note that, by continuity of the map $\psi^I_p \colon S^p_I \widehat{\otimes} S^{p^*}_I \to S^1_I$, the sequence $\big(\sum_{l=1}^{m_j} C_{l,j} * D_{l,j}\big)_{j \geqslant 1}$ is convergent. Now, we have

$$\begin{aligned}
\psi_p^{I\times I}\beta\alpha(E) &= \psi_p^{I\times I}\beta\alpha\left(\left(\lim_{i\to+\infty}\sum_{k=1}^{n_i} A_{k,i}\otimes B_{k,i}\right)\otimes\left(\lim_{j\to+\infty}\sum_{l=1}^{m_j} C_{l,j}\otimes D_{l,j}\right)\right)\\
&= \lim_{i\to+\infty}\lim_{j\to+\infty}\sum_{k=1}^{n_i}\sum_{l=1}^{m_j}\psi_p^{I\times I}\beta\alpha(A_{k,i}\otimes B_{k,i}\otimes C_{l,j}\otimes D_{l,j})\\
&= \lim_{i\to+\infty}\lim_{j\to+\infty}\sum_{k=1}^{n_i}\sum_{l=1}^{m_j}\psi_p^{I\times I}(A_{k,i}\otimes C_{l,j}\otimes B_{k,i}\otimes D_{l,j})\\
&= \lim_{i\to+\infty}\lim_{j\to+\infty}\sum_{k=1}^{n_i}\sum_{l=1}^{m_j}(A_{k,i}\otimes C_{l,j})*(B_{k,i}\otimes D_{l,j})\\
&= \lim_{i\to+\infty}\lim_{j\to+\infty}\sum_{k=1}^{n_i}\sum_{l=1}^{m_j}(A_{k,i}*B_{k,i})\otimes(C_{l,j}*D_{l,j})\\
&= \left(\lim_{i\to+\infty}\sum_{k=1}^{n_i} A_{k,i}*B_{k,i}\right)\otimes\left(\lim_{j\to+\infty}\sum_{l=1}^{m_j} C_{l,j}*D_{l,j}\right)\\
&= 0 \qquad \text{by (3.1).}
\end{aligned}$$

We prove that $S_I^p\widehat{\otimes}S_I^{p^*}\otimes\mathrm{Ker}\,(\psi_p^I)\subset\mathrm{Ker}\,(\psi_p^{I\times I}\beta\alpha)$ by a similar computation. The proof is complete. ■

Now, we define the map $V\colon\mathbb{M}_I^{\mathrm{fin}}\otimes\mathbb{M}_I^{\mathrm{fin}}\to\mathbb{M}_I^{\mathrm{fin}}\otimes\mathbb{M}_I^{\mathrm{fin}}$ by $V(e_{ij}\otimes e_{kl})=\delta_{kl}\ e_{ik}\otimes e_{kj}$.

Proposition 3.2 *With respect to trace duality, the map* $W\colon\mathbb{M}_I^{\mathrm{fin}}\otimes\mathbb{M}_I^{\mathrm{fin}}\to\mathbb{M}_I^{\mathrm{fin}}\otimes\mathbb{M}_I^{\mathrm{fin}}$ *defined by*

$$W(e_{ij}\otimes e_{kl})=\delta_{jk}\ e_{il}\otimes e_{jj}$$

is the dual map of V*. Moreover, the map* V *induces a partial isometry* $V\colon S_I^2\otimes_2 S_I^2\to S_I^2\otimes_2 S_I^2$*.*

Proof : For all $i,j,k,l,r,s,t,u\in I$, we have

$$\begin{aligned}
\mathrm{Tr}\,\Big(V(e_{ij}\otimes e_{kl})(e_{rs}\otimes e_{tu})^T\Big) &= \delta_{kl}\,\mathrm{Tr}\,\Big((e_{ik}\otimes e_{kj})(e_{rs}^T\otimes e_{tu}^T)\Big)\\
&= \delta_{kl}\,\mathrm{Tr}\,(e_{ik}e_{rs}^T)\,\mathrm{Tr}\,(e_{kj}e_{tu}^T)\\
&= \delta_{klst}\delta_{ir}\delta_{ju}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tr}\,\Big((e_{ij}\otimes e_{kl})\big(W(e_{rs}\otimes e_{tu})\big)^T\Big) &= \delta_{st}\,\mathrm{Tr}\,\Big((e_{ij}\otimes e_{kl})(e_{ru}\otimes e_{ss})^T\Big)\\
&= \delta_{st}\,\mathrm{Tr}\,(e_{ij}e_{ru}^T)\,\mathrm{Tr}\,(e_{kl}e_{ss}^T)\\
&= \delta_{klst}\delta_{ir}\delta_{ju}.
\end{aligned}$$

We conclude that W is the dual map of V. The fact that V induces a partial isometry is clear. ■

Proposition 3.3 *Suppose $1 \leqslant p \leqslant \infty$. The linear maps $V \colon \mathbb{M}_I^{\mathrm{fin}} \otimes \mathbb{M}_I^{\mathrm{fin}} \to \mathbb{M}_I^{\mathrm{fin}} \otimes \mathbb{M}_I^{\mathrm{fin}}$ and $W \colon \mathbb{M}_I^{\mathrm{fin}} \otimes \mathbb{M}_I^{\mathrm{fin}} \to \mathbb{M}_I^{\mathrm{fin}} \otimes \mathbb{M}_I^{\mathrm{fin}}$ admit completely contractive extensions $V \colon S_I^p(S_I^p) \to S_I^p(S_I^p)$ and $W \colon S_I^p(S_I^p) \to S_I^p(S_I^p)$.*

Proof : We first prove that V and W admit completely contractive extensions from $S_I^\infty(S_I^\infty)$ into $S_I^\infty(S_I^\infty)$. Suppose that $B = \sum_{i,j,k,l \in J} b_{ijkl} \otimes e_{ij} \otimes e_{kl} \in \mathbb{M}_{\mathrm{fin}} \otimes \mathbb{M}_I^{\mathrm{fin}} \otimes \mathbb{M}_I^{\mathrm{fin}}$ with $J \in \mathcal{P}_f(I)$ and $b_{ijkl} \in \mathbb{M}_{\mathrm{fin}}$ for all $i,j,k,l \in J$. Note that the matrix $U = \displaystyle\sum_{r,s \in J} e_{rs} \otimes e_{sr}$ of $S_J^\infty(S_J^\infty)$ is unitary. Then we have

$$
\begin{aligned}
\big\|(Id_{S^\infty} \otimes V)(B)\big\|_{S^\infty(S_I^\infty(S_I^\infty))} &= \Bigg\| \sum_{i,j,k \in J} b_{ijkk} \otimes e_{ik} \otimes e_{kj} \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| \Bigg(I_{S^\infty} \otimes \Bigg(\sum_{r,s \in J} e_{rs} \otimes e_{sr} \Bigg) \Bigg) \Bigg(\sum_{i,j,k \in J} b_{ijkk} \otimes e_{ik} \otimes e_{kj} \Bigg) \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| \sum_{r,s,i,j,k \in J} b_{ijkk} \otimes e_{rs} e_{ik} \otimes e_{sr} e_{kj} \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| \sum_{i,j,k \in J} b_{ijkk} \otimes e_{kk} \otimes e_{ij} \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| \sum_{k \in J} e_{kk} \otimes \Bigg(\sum_{i,j \in I} b_{ijkk} \otimes e_{ij} \Bigg) \Bigg\|_{S_I^\infty(S^\infty(S_I^\infty))} \\
&= \max_{k \in J} \Bigg\| \sum_{i,j \in I} b_{ijkk} \otimes e_{ij} \Bigg\|_{S^\infty(S_I^\infty)} \\
&\leqslant \|B\|_{S^\infty(S_I^\infty(S_I^\infty))} \quad \text{(submatrices)}
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|(Id_{S^\infty} \otimes W)(B)\big\|_{S^\infty(S_I^\infty(S_I^\infty))} &= \Bigg\| \sum_{i,j,l \in J} b_{ijjl} \otimes e_{il} \otimes e_{jj} \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| (I_{S^\infty} \otimes U) \Bigg(\sum_{i,j,l \in J} b_{ijjl} \otimes e_{il} \otimes e_{jj} \Bigg) (I_{S^\infty} \otimes U) \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| \sum_{r,s,i,j,l,t,u \in J} b_{ijjl} \otimes e_{rs} e_{il} e_{tu} \otimes e_{sr} e_{jj} e_{ut} \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \Bigg\| \sum_{i,j,l \in J} b_{ijjl} \otimes e_{jj} \otimes e_{il} \Bigg\|_{S^\infty(S_I^\infty(S_I^\infty))}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \sum_{j \in J} e_{jj} \otimes \left(\sum_{i,l \in J} b_{ijjl} \otimes e_{il} \right) \right\|_{S_I^\infty(S^\infty(S_I^\infty))} \\
&= \max_{j \in J} \left\| \sum_{i,l \in J} b_{ijjl} \otimes e_{il} \right\|_{S^\infty(S_I^\infty)} \\
&\leqslant \left\| \sum_{i,j,k,l \in J} b_{ijkl} \otimes e_{kj} \otimes e_{il} \right\|_{S^\infty(S_I^\infty(S_I^\infty))} \qquad \text{(submatrices)} \\
&= \left\| \left(I_{S^\infty} \otimes \left(\sum_{r,s \in J} e_{rs} \otimes e_{sr} \right) \right) \left(\sum_{i,j,k,l \in J} b_{ijkl} \otimes e_{kj} \otimes e_{il} \right) \right\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \left\| \sum_{r,s,i,j,k \in J} b_{ijkl} \otimes e_{rs} e_{kj} \otimes e_{sr} e_{il} \right\|_{S^\infty(S_I^\infty(S_I^\infty))} \\
&= \|B\|_{S^\infty(S_I^\infty(S_I^\infty))}.
\end{aligned}
$$

Then we deduce the claim. Hence, by duality, the linear maps $V^*\colon S_1^I(S_1^I) \to S_1^I(S_1^I)$ and $W^*\colon S_1^I(S_1^I) \to S_1^I(S_1^I)$ are completely contractive. Moreover, we know that $W = V^*$. By interpolation between $p=1$ and $p=\infty$, we obtain that the maps $V\colon S_I^p(S_I^p) \to S_I^p(S_I^p)$ and $W\colon S_I^p(S_I^p) \to S_I^p(S_I^p)$ are completely contractive. ■

Now, we define the linear map

$$
\begin{array}{cccc}
\Delta\colon & \mathbb{M}_I & \longrightarrow & \mathbb{M}_{I \times I} \\
 & A & \longmapsto & [a_{ts}\delta_{ur}]_{(t,r),(u,s) \in I \times I}.
\end{array}
$$

Proposition 3.4 *Let $1 \leqslant p \leqslant \infty$. Suppose that $M_A\colon S_I^p \to S_I^p$ is a completely bounded Schur multiplier on S_I^p associated with a matrix A of $\mathbb{M}_I$. Then the map $V(M_A \otimes Id_{S_I^p})W$ is a bounded Schur multiplier on $S_I^p(S_I^p)$. Its associated matrix is $\Delta(A)$.*

Proof : If $i,j,k,l \in I$ and $M_A \in \mathfrak{M}_{p,cb}^I$, we have

$$
\begin{aligned}
M_{\Delta(A)}(e_{ij} \otimes e_{kl}) &= \left([a_{ts}\delta_{ur}]_{(t,r),(u,s) \in I \times I} \right) * \left([\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s) \in I \times I} \right) \\
&= \delta_{jk} a_{il} \left([\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s) \in I \times I} \right) \\
&= \delta_{jk} a_{il} e_{ik} \otimes e_{kl}
\end{aligned}
$$

and

$$
\begin{aligned}
V(M_A \otimes Id_{S_I^p})W(e_{ij} \otimes e_{kl}) &= \delta_{jk} V(M_A \otimes Id_{S_I^p})(e_{il} \otimes e_{jj}) \\
&= \delta_{jk} a_{il} V(e_{il} \otimes e_{kk}) \\
&= \delta_{jk} a_{il} e_{ik} \otimes e_{kl}.
\end{aligned}
$$

■

Recall that, for all operator spaces E and F, the map $R\otimes T \mapsto R\otimes T$ is completely contractive from $CB(E)\widehat{\otimes}CB(F)$ into $CB(E\otimes_{\min}F)$ and from $CB(E)\widehat{\otimes}CB(F)$ into $CB(E\widehat{\otimes}F)$ (see [14, Proposition 5.11]).

Proposition 3.5 *Suppose* $1 \leqslant p \leqslant \infty$. *Let* I, J *be any sets. The map*

$$\begin{array}{ccc} CB(S_I^p) & \longrightarrow & CB(S_I^p(S_J^p)) \\ T & \longmapsto & T\otimes Id_{S_J^p} \end{array}$$

is a complete contraction.

Proof : By definition, we have $S_J^\infty(S_I^p) = S_J^\infty \otimes_{\min} S_I^p$ and $S_J^1(S_I^p) = S_J^1\widehat{\otimes}S_I^p$ completely isometrically. Then we obtain two complete contractions

$$\begin{array}{ccccc} CB(S_I^p) & \longrightarrow & CB(S_J^\infty)\widehat{\otimes}CB(S_I^p) & \longrightarrow & CB(S_J^\infty(S_I^p)) \\ T & \longmapsto & Id_{S_J^\infty}\otimes T & \longmapsto & Id_{S_J^\infty}\otimes T \end{array}$$

and

$$\begin{array}{ccccc} CB(S_I^p) & \longrightarrow & CB(S_J^1)\widehat{\otimes}CB(S_I^p) & \longrightarrow & CB(S_J^1(S_I^p)) \\ T & \longmapsto & Id_{S_J^1}\otimes T & \longmapsto & Id_{S_J^1}\otimes T. \end{array}$$

By interpolation, we obtain a completely contractive map

$$CB(S_I^p) \to \Big(CB(S_J^\infty(S_I^p)), CB(S_J^1(S_I^p))\Big)_{\frac{1}{p}}.$$

We conclude by composing with the complete contraction

$$\Big(CB(S_J^\infty(S_I^p)), CB(S_J^1(S_I^p))\Big)_{\frac{1}{p}} \to CB(S_J^p(S_I^p))$$

and by using the Fubini's theorem (see [99, Theorem 1.9]). ■

Remark 3.6 *It is easy to see that this map is completely isometric.*

The next theorem is the principal result of this paper.

Theorem 3.7 *Suppose* $1 \leqslant p < \infty$. *The space* $\mathfrak{R}_{p,cb}^I$ *equipped with the usual matricial product is a completely contractive Banach algebra. More precisely, if* A *and* B *are matrices of* $\mathfrak{R}_{p,cb}^I$ *and* $i, j \in I$, *the limit* $\lim_J \sum_{k\in J} a_{ik}b_{kj}$ *exists. Moreover, the matrix* $A.B$ *of* $\mathbb{M}_I$ *defined by* $[A.B]_{ij} = \lim_J \sum_{k\in J} a_{ik}b_{kj}$ *belongs to* $\mathfrak{R}_{p,cb}^I$. *Finally, the map*

$$\begin{array}{ccc} \mathfrak{R}_{p,cb}^I\widehat{\otimes}\mathfrak{R}_{p,cb}^I & \longrightarrow & \mathfrak{R}_{p,cb}^I \\ A\otimes B & \longmapsto & AB \end{array}$$

is completely contractive.

Proof : We have already seen that it suffices to prove the theorem with $1 < p < \infty$. If $M_A \in \mathfrak{M}^I_{p,cb}$, by Proposition 3.4, we have the following commutative diagram

$$\begin{tikzcd}
S^p_I(S^p_I) \arrow[r, "M_{\Delta(A)}"] \arrow[d, "W"'] & S^p_I(S^p_I) \\
S^p_I(S^p_I) \arrow[r, "M_A \otimes Id_{S^p_I}"'] & S^p_I(S^p_I). \arrow[u, "V"']
\end{tikzcd}$$

By Proposition 3.5, the map $M_A \mapsto M_A \otimes Id_{S^p_I}$ is completely contractive from $\mathfrak{M}^I_{p,cb}$ into $\mathfrak{M}^{I\times I}_{p,cb}$. Moreover it is easy to see that this map is w*-continuous. Since $S^p_I(S^p_I)$ is reflexive, by Lemma 2.1 and by composition, the map $M_A \mapsto M_{\Delta(A)}$ from $\mathfrak{M}^I_{p,cb}$ into $\mathfrak{M}^{I\times I}_{p,cb}$ is a complete contraction and is w*-continuous. We denote by $\Delta_* \colon \mathfrak{R}^{I\times I}_{p,cb} \to \mathfrak{R}^I_{p,cb}$ its preadjoint. Now, by Lemma 2.5, we have for all $i, j \in I$ and for all matrices A, B of $\mathbb{M}^{\text{fin}}_I$

$$\begin{aligned}
\big[\Delta_*(A\otimes B)\big]_{ij} &= \Big\langle M_{e_{ij}}, \Delta_*(A\otimes B)\Big\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}} \\
&= \Big\langle M_{\Delta(e_{ij})}, A\otimes B\Big\rangle_{\mathfrak{M}^{I\times I}_{p,cb},\mathfrak{R}^{I\times I}_{p,cb}} \\
&= \Big\langle M_{[\delta_{it}\delta_{js}\delta_{ur}]_{(t,r),(u,s)\in I\times I}}, [a_{tu}b_{rs}]_{(t,r),(u,s)\in I\times I}\Big\rangle_{\mathfrak{M}^{I\times I}_{p,cb},\mathfrak{R}^{I\times I}_{p,cb}} \\
&= \lim_J \sum_{r\in J} a_{ir}b_{rj} \\
&= [A.B]_{ij}.
\end{aligned}$$

Thus we conclude that, if $A, B \in \mathbb{M}^{\text{fin}}_I$, we have $\Delta_*(A\otimes B) = AB$. By Proposition 3.1 and by density of $\mathbb{M}^{\text{fin}}_I \otimes \mathbb{M}^{\text{fin}}_I$ in $\mathfrak{R}^I_{p,cb}\widehat{\otimes}\mathfrak{R}^I_{p,cb}$, we deduce that the map

$$\begin{array}{ccccc}
\mathbb{M}^{\text{fin}}_I \otimes \mathbb{M}^{\text{fin}}_I & \longrightarrow & \mathfrak{R}^{I\times I}_{p,cb} & \xrightarrow{\Delta_*} & \mathfrak{R}^I_{p,cb} \\
A\otimes B & \longmapsto & A\otimes B & \longmapsto & AB
\end{array}$$

admits a unique bounded extension from $\mathfrak{R}^I_{p,cb}\widehat{\otimes}\mathfrak{R}^I_{p,cb}$ into $\mathfrak{R}^I_{p,cb}$. Moreover, this map is completely contractive. Finally, we complete the proof by a straightforward approximation argument using Lemma 2.5. ■

Remark 3.8 *We do not know if the space $\mathfrak{R}^I_p$ equipped with the usual matricial product is a Banach algebra. The Banach space analogue of Proposition 3.5 is false. It is the reason which explains that* *the method does not work for* $\mathfrak{R}_p^I$. *However, note that if* $\mathfrak{M}_p^I = \mathfrak{M}_{p,cb}^I$ *isometrically we have* $\mathfrak{R}_p^I = \mathfrak{R}_{p,cb}^I$ *isometrically. For* $1 < p < \infty$, $p \neq 2$ *the equality* $\mathfrak{M}_p^I = \mathfrak{M}_{p,cb}^I$ *is a classical open question.*

4 Schur product

In this section, we replace the matricial product by the Schur product. First, it is easy to show the following proposition.

Proposition 4.1 *Suppose* $1 \leqslant p < \infty$. *The Banach space* $\mathfrak{R}_p^I$ *equipped with the Schur product is a commutative Banach algebra.*

Proof : It suffices to use the equality (2.2) and the fact that S_I^p equipped with the Schur product is a Banach algebra (see [13, page 225]). ■

Now we will show the completely bounded analogue of this proposition. We define the pointwise product

$$\begin{array}{rccc} P: & \ell_I^1 \widehat{\otimes} \ell_I^1 & \longrightarrow & \ell_I^1 \\ & e_i \otimes e_j & \longmapsto & \delta_{ij} e_i. \end{array}$$

This map is well-defined and is completely contractive (see [13, page 211]). Then, by tensoring, we obtain a completely contractive map

$$P \otimes P \colon (\ell_I^1 \widehat{\otimes} \ell_I^1) \otimes_h (\ell_I^1 \widehat{\otimes} \ell_I^1) \to \ell_I^1 \otimes_h \ell_I^1. \tag{4.1}$$

By [38, Theorem 6.1], the map

$$\begin{array}{ccc} (\ell_I^\infty \overline{\otimes} \ell_I^\infty) \otimes_{\sigma h} (\ell_I^\infty \overline{\otimes} \ell_I^\infty) & \longrightarrow & (\ell_I^\infty \otimes_{\sigma h} \ell_I^\infty) \overline{\otimes} (\ell_I^\infty \otimes_{\sigma h} \ell_I^\infty) \\ a \otimes b \otimes c \otimes d & \longmapsto & a \otimes c \otimes b \otimes d \end{array}$$

is completely contractive. Moreover, by [38, (5.23)], we have the following commutative diagram

$$\begin{tikzcd} (\ell_I^\infty \overline{\otimes} \ell_I^\infty) \otimes_{\sigma h} (\ell_I^\infty \overline{\otimes} \ell_I^\infty) \arrow[r] & (\ell_I^\infty \otimes_{\sigma h} \ell_I^\infty) \overline{\otimes} (\ell_I^\infty \otimes_{\sigma h} \ell_I^\infty) \\ (\ell_I^\infty \overline{\otimes} \ell_I^\infty) \otimes_{eh} (\ell_I^\infty \overline{\otimes} \ell_I^\infty) \arrow[r] \arrow[u, hook] & (\ell_I^\infty \otimes_{eh} \ell_I^\infty) \overline{\otimes} (\ell_I^\infty \otimes_{eh} \ell_I^\infty). \arrow[u, hook] \end{tikzcd}$$

By [38, Theorem 4.2], [38, Theorem 5.3] and by duality, we deduce that the map

$$\begin{array}{ccc} (\ell_I^1 \otimes_h \ell_I^1) \widehat{\otimes} (\ell_I^1 \otimes_h \ell_I^1) & \longrightarrow & (\ell_I^1 \widehat{\otimes} \ell_I^1) \otimes_h (\ell_I^1 \widehat{\otimes} \ell_I^1) \\ a \otimes b \otimes c \otimes d & \longmapsto & a \otimes c \otimes b \otimes d \end{array}$$

is well-defined and completely contractive. Composing this map and (4.1), we deduce a completely contractive map

$$\begin{array}{ccc}(\ell^1_I \otimes_h \ell^1_I)\widehat{\otimes}(\ell^1_I \otimes_h \ell^1_I) & \longrightarrow & \ell^1_I \otimes_h \ell^1_I \\ a \otimes b \otimes c \otimes d & \longmapsto & P(a \otimes c) \otimes P(b \otimes d).\end{array}$$

With the identification $\mathfrak{R}^I_{1,cb} = \ell^1_I \otimes_h \ell^1_I$, we obtain a completely contractive map

$$\begin{array}{ccc}\mathfrak{R}^I_{1,cb}\widehat{\otimes}\mathfrak{R}^I_{1,cb} & \longrightarrow & \mathfrak{R}^I_{1,cb} \\ A \otimes B & \longmapsto & A * B.\end{array}$$

This means that $\mathfrak{R}^I_{1,cb}$ equipped with the Schur product is a completely contractive Banach algebra. Now, recall that we have $\mathfrak{R}^I_{2,cb} = \ell^1_I \widehat{\otimes} \ell^1_I$ completely isometrically. Then, by a similar argument, $\mathfrak{R}^I_{2,cb}$ equipped with the Schur product is also a completely contractive Banach algebra. We will use a strategy similar to that used in the proof of Theorem 3.7 for other values of p.

We start by defining the Schur multiplier $M_E \colon S^p_I(S^p_I) \to S^p_I(S^p_I)$ associated with the matrix $E = [\delta_{rt}\delta_{su}]_{(t,r),(u,s)\in I\times I}$ of $\mathbb{M}_{I\times I}$. It is not difficult to see that M_E is a completely positive contraction. Note that, for all $i,j,k,l \in I$, we have

$$\begin{aligned} M_E(e_{ij} \otimes e_{kl}) &= \Big([\delta_{rt}\delta_{su}]_{(t,r),(u,s)\in I\times I}\Big) * \Big([\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s)\in I\times I}\Big) \\ &= \delta_{ik}\delta_{jl}[\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s)\in I\times I} \\ &= \delta_{ik}\delta_{jl} e_{ij} \otimes e_{kl}. \end{aligned}$$

Now, we define the linear map

$$\begin{array}{rccc}\eta\colon & \mathbb{M}_I & \longrightarrow & \mathbb{M}_{I\times I} \\ & A & \longmapsto & [a_{rs}\delta_{rt}\delta_{su}]_{(t,r),(u,s)\in I\times I}.\end{array}$$

Proposition 4.2 *Let $1 \leqslant p \leqslant \infty$. Suppose that $M_A \colon S^p_I \to S^p_I$ is a completely bounded Schur multiplier on S^p_I associated with a matrix A. Then the map $M_E(M_A \otimes Id_{S^p_I})M_E$ is a bounded Schur multiplier on $S^p_I(S^p_I)$. Its associated matrix is $\eta(A)$.*

Proof : If $i,j,k,l \in I$ and $M_A \in \mathfrak{M}^I_{p,cb}$, we have

$$\begin{aligned} M_{\eta(A)}(e_{ij} \otimes e_{kl}) &= \Big([a_{rs}\delta_{rt}\delta_{su}]_{(t,r),(u,s)\in I\times I}\Big) * \Big([\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s)\in I\times I}\Big) \\ &= \delta_{ik}\delta_{jl}a_{ij}[\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s)\in I\times I} \\ &= \delta_{ik}\delta_{jl}a_{ij} e_{ij} \otimes e_{kl} \end{aligned} \tag{4.2}$$

and

$$M_E(M_A \otimes Id_{S^p_I})M_E(e_{ij} \otimes e_{kl}) = \delta_{ik}\delta_{jl} M_E(M_A \otimes Id_{S^p_I})(e_{ij} \otimes e_{kl})$$

$$= \delta_{ik}\delta_{jl}a_{ij}e_{ij} \otimes e_{kl}.$$

■

Theorem 4.3 *Suppose* $1 \leqslant p < \infty$. *The space* $\mathfrak{R}^I_{p,cb}$ *equipped with the Schur product is a commutative completely contractive Banach algebra.*

Proof : We have already seen that it suffices to prove the theorem with $1 < p < \infty$. If $M_A \in \mathfrak{M}^I_{p,cb}$, by Proposition 4.2, we have the following commutative diagram

$$\begin{tikzcd} S^p_I(S^p_I) \arrow[r, "M_{\eta(A)}"] \arrow[d, "M_E"'] & S^p_I(S^p_I) \\ S^p_I(S^p_I) \arrow[r, "M_A \otimes Id_{S^p_I}"'] & S^p_I(S^p_I). \arrow[u, "M_E"'] \end{tikzcd}$$

We have already seen that the map $M_A \mapsto M_A \otimes Id_{S^p_I}$ is completely contractive from $\mathfrak{M}^I_{p,cb}$ into $\mathfrak{M}^{I\times I}_{p,cb}$ and w*-continuous. Since $S^p_I(S^p_I)$ is reflexive, by Lemma 2.1 and by composition, the map $M_A \mapsto M_{\eta(A)}$ from $\mathfrak{M}^I_{p,cb}$ into $\mathfrak{M}^{I\times I}_{p,cb}$ is a complete contraction and is w*-continuous.

We denote by $\eta_* \colon \mathfrak{R}^{I\times I}_{p,cb} \to \mathfrak{R}^I_{p,cb}$ its preadjoint. Now, by Lemma 2.5, we have for all $i, j \in I$ and for all matrices A, B of $\mathbb{M}^{\text{fin}}_I$

$$\begin{aligned} \big[\eta_*(A \otimes B)\big]_{ij} &= \Big\langle M_{e_{ij}}, \eta_*(A \otimes B)\Big\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}} \\ &= \Big\langle M_{\eta(e_{ij})}, A \otimes B\Big\rangle_{\mathfrak{M}^{I\times I}_{p,cb},\mathfrak{R}^{I\times I}_{p,cb}} \\ &= \Big\langle M_{[\delta_{ir}\delta_{js}\delta_{rt}\delta_{su}]_{(t,r),(u,s)\in I\times I}}, \big[a_{tu}b_{rs}\big]_{(t,r),(u,s)\in I\times I}\Big\rangle_{\mathfrak{M}^{I\times I}_{p,cb},\mathfrak{R}^{I\times I}_{p,cb}} \\ &= a_{ij}b_{ij} \\ &= [A * B]_{ij}. \end{aligned}$$

Thus we conclude that if $A, B \in \mathbb{M}^{\text{fin}}_I$ we have $\eta_*(A \otimes B) = A * B$. By Proposition 3.1 and by density of $\mathbb{M}^{\text{fin}}_I \otimes \mathbb{M}^{\text{fin}}_I$ in $\mathfrak{R}^I_{p,cb}\widehat{\otimes}\mathfrak{R}^I_{p,cb}$, we deduce that the map

$$\begin{array}{ccccc} \mathbb{M}^{\text{fin}}_I \otimes \mathbb{M}^{\text{fin}}_I & \longrightarrow & \mathfrak{R}^{I\times I}_{p,cb} & \stackrel{\eta_*}{\longrightarrow} & \mathfrak{R}^I_{p,cb} \\ A \otimes B & \longmapsto & A \otimes B & \longmapsto & A * B \end{array}$$

admits a unique bounded extension from $\mathfrak{R}^I_{p,cb}\widehat{\otimes}\mathfrak{R}^I_{p,cb}$ into $\mathfrak{R}^I_{p,cb}$. Moreover, this map is completely contractive. Finally, we complete the proof by a straightforward approximation argument with Lemma 2.5. ■

Now, we will give a more simple proof of this theorem. It is easy to see that η induces a completely isometric map $\eta\colon S_I^p \to S_I^p(S_I^p)$. Moreover, by the computation (4.2), its range is clearly 1-completely complemented by $M_E\colon S_I^p(S_I^p) \to S_I^p(S_I^p)$. We denote by $\eta^{-1}\colon \eta(S_I^p(S_I^p)) \to S_I^p$ the inverse map of η. For all $B \in \eta(S_I^p(S_I^p))$, we have $\eta^{-1}(B) = \big[b_{(r,r),(s,s)}\big]_{r,s\in I}$. Finally, for all $i,j,k,l \in I$ we have

$$\begin{aligned}
\eta M_A \eta^{-1} M_E(e_{ij} \otimes e_{kl}) &= \delta_{ik}\delta_{jl}\eta M_A \eta^{-1}(e_{ij} \otimes e_{kl})\\
&= \delta_{ik}\delta_{jl}\eta M_A \eta^{-1}\Big([\delta_{it}\delta_{ju}\delta_{kr}\delta_{ls}]_{(t,r),(u,s)\in I\times I}\Big)\\
&= \delta_{ik}\delta_{jl}\eta M_A\Big([\delta_{ir}\delta_{js}\delta_{kr}\delta_{ls}]_{r,s\in I}\Big)\\
&= \delta_{ik}\delta_{jl}a_{ij}\eta\Big([\delta_{ir}\delta_{js}\delta_{kr}\delta_{ls}]_{r,s\in I}\Big)\\
&= \delta_{ik}\delta_{jl}a_{ij}e_{ij} \otimes e_{kl}\\
&= M_{\eta(A)}(e_{ij} \otimes e_{kl})
\end{aligned}$$

where we have used the computation (4.2) in the last equality.

Hence we have the following commutative diagram

$$\begin{tikzcd}
S_I^p(S_I^p) \arrow[rr, "M_{\eta(A)}"] \arrow[d, "M_E"'] & & S_I^p(S_I^p) \\
\eta(S_I^p(S_I^p)) \arrow[d, "\eta^{-1}"'] & & \\
S_I^p \arrow[rr, "M_A"'] & & S_I^p. \arrow[uu, "\eta"']
\end{tikzcd}$$

We conclude with an argument similar to that used in the proof of Theorem 4.3.

5 Isometric multipliers

The next result is the noncommutative version of a theorem of Parrott [85] and Strichartz [111] which states that every isometric Fourier multiplier on $L^p(G)$ for $1 \leqslant p \leqslant \infty$, $p \neq 2$, is a scalar multiple of an operator induced by a translation.

Theorem 5.1 *Suppose $1 \leqslant p \leqslant \infty$, $p \neq 2$. An isometric Schur multiplier on S_I^p is defined by a matrix $[a_i b_j]$ with $a_i, b_j \in \mathbb{T}$.*

Proof : Suppose that M_C is an isometric Schur multiplier on the Banach space S_I^p defined by a matrix C. First, we observe that M_C is onto. Indeed, for all $i,j \in I$, we have $M_C(e_{ij}) = c_{ij}e_{ij}$. Then $c_{ij} \neq 0$ since M_C is one-to-one. Consequently e_{ij} belongs to the range of M_C. By density, we conclude that M_C is onto.

Now we use the theorem of Arazy [5] which describes the onto isometries on S_I^p. Then there exists two unitaries $U = [u_{ij}]$ and $V = [v_{ij}]$ of $B(\ell_I^2)$ satisfying for all $A \in S_I^p$

$$C * A = UAV \quad \text{or} \quad C * A = UA^TV.$$

Examine the first case, we have for all $k,l \in I$

$$Ue_{kl}V = C * e_{kl}.$$

Hence, for all $i,j \in I$, we have the equality

$$[Ue_{kl}V]_{ij} = [C * e_{kl}]_{ij}.$$

Since

$$[Ue_{kl}V]_{ij} = u_{ik}v_{lj}$$

we have

$$u_{ik}v_{lj} = \begin{cases} c_{kl} & \text{if } i = k \text{ and } j = l \\ 0 & \text{if } i \neq k \text{ or if } j \neq l. \end{cases}$$

Then $u_{kk}v_{ll} = c_{kl}$. Each c_{kl} is non null since the image of each e_{kl} by the map M_C cannot be null. Then, for all k and all l, we have $u_{kk} \neq 0$ and $v_{ll} \neq 0$. And for $i \neq k$, we have $u_{ik}v_{ll} = 0$. Then if $i \neq k$, we have $u_{ik} = 0$. Now if $j \neq l$, we have $u_{kk}v_{lj} = 0$. Then if $j \neq l$, we have $v_{lj} = 0$. Finally, for all $i,j \in I$, we define the complex numbers $a_i = u_{ii}$ and $b_j = v_{jj}$. Since the diagonal matrices U and V are unitaries, we have $a_i, b_j \in \mathbb{T}$. Thus we have the required form.

Examine the second case. We have for all $k,l \in I$

$$Ue_{lk}V = C * e_{kl}.$$

We deduce that, for all $i,j,k,l \in I$, we have

$$[Ue_{lk}V]_{ij} = [C * e_{kl}]_{ij}.$$

Since

$$[Ue_{lk}V]_{ij} = u_{il}v_{kj}$$

we obtain $u_{kl}v_{kl} = c_{kl}$ and $u_{il}v_{kj} = 0$ if $i \neq k$ or if $j \neq l$. Each c_{kl} is non null since the image of each e_{kl} by the map M_C cannot be null. Then for all k,l we have $u_{kl} \neq 0$ and $v_{kl} \neq 0$. Thus the second case is absurd (if $\mathrm{card}(I) > 1$).

The converse is straightforward. ■

Remark 5.2 *It is easy to see that an isometric Schur multiplier on S^2_I is defined by a matrix $[a_{ij}]$ with $a_{ij} \in \mathbb{T}$.*

The next result is the noncommutative version of a theorem of Figà-Talamanca [41] which states that the space of bounded Fourier multipliers is the closure in the weak operator topology of the span of translation operators.

Theorem 5.3 *Suppose $1 \leqslant p < \infty$.*

1. *The space $\mathfrak{M}^I_{p,cb}$ of completely bounded Schur multipliers on S^p_I is the closure of the span of isometric Schur multipliers in the weak* topology and in the weak operator topology.*
2. *The space $\mathfrak{M}^I_p$ of bounded Schur multipliers on S^p_I is the closure of the span of isometric Schur multipliers in the weak* topology and in the weak operator topology.*

Proof : We will only prove the part 1. The proof of the part 2 is similar.

It is easy to see that an isometric Schur multiplier on S^p_I is completely isometric. This fact allows us to consider the span of isometric Schur multipliers in $\mathfrak{M}^I_{p,cb}$. Let C be a matrix of $\mathfrak{R}^I_{p,cb}$. Suppose that C belongs to the orthogonal of the set of isometric Schur multipliers. Thus, we have for any isometric multiplier $M_{[a_ib_j]}$ (with $a_i, b_j \in \mathbb{T}$)

$$\begin{aligned} 0 &= \Big\langle M_{[a_ib_j]}, C\Big\rangle_{\mathfrak{M}^I_{p,cb},\mathfrak{R}^I_{p,cb}} \\ &= \lim_J \sum_{i,j\in J} a_ib_jc_{ij}. \end{aligned}$$

Let i_0, j_0 be elements of I. Now, we choose the a_i's, b_j's, a'_i's and b'_j's such that $a_i = b_j = 1$ for all $i, j \in I$, $a'_i = -1$ if $i \neq i_0$, $a'_{i_0} = 1$, $b'_j = -1$ if $j \neq j_0$ and $b'_{j_0} = 1$. Then, we have

$$\begin{aligned} 0 &= \lim_J \sum_{i,j\in J} a_ib_jc_{ij} + \lim_J \sum_{i,j\in J} a_ib'_jc_{ij} + \lim_J \sum_{i,j\in J} a'_ib_jc_{ij} + \lim_J \sum_{i,j\in J} a'_ib'_jc_{ij} \\ &= \lim_J \sum_{i,j\in J} (a_i + a'_i)(b_j + b'_j)c_{ij} \\ &= 4c_{i_0j_0}. \end{aligned}$$

Hence $c_{i_0j_0} = 0$. It follows that $C = 0$. Then, we deduce that the space $\mathfrak{M}^I_{p,cb}$ of completely bounded Schur multipliers is the closure of the span of isometric Schur multipliers in the weak* topology. Moreover, this topology is more finer that the weak operator topology. Thus, the proof is complete. ■

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Estimations de normes dans les espaces L^p non commutatifs et applications

Résumé

Cette thèse présente quelques résultats d'analyse sur les espaces L^p le plus souvent non commutatifs. La première partie exhibe de large classes de contractions sur des espaces L^p non commutatifs qui vérifient l'analogue non commutatif de la conjecture de Matsaev. De plus, cette partie fournit une comparaison entre certaines normes apparaissant naturellement dans ce domaine. La deuxième partie traite des fonctions carrées. Le premier résultat principal énonce que si T est un opérateur R-Ritt sur un espace L^p alors les fonctions carrées associées sont équivalentes. Le second résultat principal est une caractérisation de certaines estimations carrées utilisant les dilatations. La troisième partie de cette thèse introduit de nouvelles fonctions carrées pour les opérateurs de Ritt définis sur des espaces L^p non commutatifs. Le résultat principal est qu'en général ces fonctions carrées ne sont pas équivalentes. Cette partie contient aussi un résultat d'équivalence entre la norme usuelle et une certaine fonction carrée. La quatrième partie introduit un analogue non commutatif de l'algèbre de Figà-Talamanca-Herz $A_p(G)$ sur le prédual naturel de l'espace d'opérateurs $\mathfrak{M}_{p,cb}$ des multiplicateurs de Schur complètement bornées sur l'espace de Schatten S^p.

Mots-clefs

Espaces L^p non commutatifs, espaces de Schatten, conjecture de Matsaev, multiplicateurs de Schur, dilatations, fonctions carrées, opérateurs de Ritt, algèbres de Figà-Talamanca-Herz, espaces d'opérateurs.

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Estimates of norms in noncommutative L^p-spaces and applications

Abstract

This thesis presents some results of analysis in L^p-spaces, especially often noncommutative. The first part exhibits large classes of contractions on noncommutative L^p-spaces which satisfy the noncommutative analogue of Matsaev's conjecture. Moreover, this part gives a comparison between various norms arising naturally from this field. The second part is devoted to square functions. The first main result states that if T is an R-Ritt operator on a L^p-space then the involved square functions are equivalent. The second principal result is a characterization of some square functions estimates in terms of dilations. In the third part of this thesis, we introduce some new square functions for Ritt operators defined on noncommutative L^p-spaces. The main result is that these square functions are generally not equivalent. This part also contains a result of equivalence between the usual norm and some special square function. The fourth part introduces a noncommutative analogue of the Figà-Talamanca-Herz algebra $A_p(G)$ on the natural predual of the operator space $\mathfrak{M}_{p,cb}$ of completely bounded Schur multipliers on the Schatten space S^p.

Keywords

Non-commutative L^p spaces, Schatten spaces, Matsaev's conjecture, Schur multipliers, dilations, square functions, Ritt operators, Figà-Talamanca-Herz algebras, operator spaces.

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